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Collocation–quadrature methods and fast summation for Cauchy singular integral equations with fixed singularities

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Basing on recent results on the stability of collocation methods applied to Cauchy singular integral equations with additional fixed singularities we give necessary and sufficient conditions for the stability of collocation–quadrature methods for such equations. These methods have the advantage that the respective system of equations has a very simple structure and allows to apply fast summation methods which results in a fast algorithm with $\mathcal{O}(n \log n)$ complexity. We present numerical results of the application of the proposed collocation–quadrature methods to the notched half plane problem of two-dimensional elasticity theory.

1. Introduction

In the paper [1] the stability of collocation methods applied to Cauchy singular integral equations with additional Mellin-type operators (cf. (2.1)) was studied. The present

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paper is a continuation of these investigations focusing on a further discretization step applied to the integral operators with fixed singularities by applying an appropriate quadrature rule. These collocation–quadrature methods, which we introduce here, have the advantage (in comparison to the pure collocation methods) that the respective system of discrete equations has a very simple structure, which enables us to apply fast summation methods resulting in a fast algorithm with $\mathcal{O}(n \log n)$ complexity. Let us remark that the methods under consideration here are based on the approximation of the unknown solution by weighted polynomials and collocating at the zeros of Chebyshev polynomials. Concerning collocation and quadrature methods, which are based on spline approximation using suitably graded meshes, we refer to Chapter 11 of [2] or [3] and the literature cited there. Here we will see that, in many situations, the set of zeros of orthogonal polynomials as the set of collocation points is already suitably graded.

The present paper is organized as follows. In Section 2 the collocation–quadrature methods, we deal with here, are introduced, Section 3 contains a short description of the C^*-algebra background, which we use for proving stability of the methods, and Section 4 shows that the operator sequences of the collocation–quadrature methods belong to the C^*-algebra defined in Section 3. In Section 5 we prove the main result, namely the stability theorem, and in Section 6 we describe the structure of the systems of discrete equations and the application of a fast summation method. In Section 7 we present numerical results obtained by applying the investigated methods to the notched half plane problem of two-dimensional elasticity theory, and the final Section 8 is devoted to the proof of convergence rates for the collocation and the collocation–quadrature methods.

2. The quadrature method

Here we consider the Cauchy singular integral equation with additional fixed singularities

$$a(x)u(x) + \frac{b(x)}{\pi \mathbf{i}} \int_{-1}^{1} \frac{u(y)\,dy}{y-x} + \sum_{k=1}^{m_-} \beta_k^- \int_{-1}^{1} \mathbf{h}_k^- \left(\frac{1+x}{1+y} \right) \frac{u(y)\,dy}{1+y}$$
$$+ \sum_{k=1}^{m_+} \beta_k^+ \int_{-1}^{1} \mathbf{h}_k^+ \left(\frac{1-x}{1-y} \right) \frac{u(y)\,dy}{1-y} = f(x), \tag{2.1}$$

$-1 < x < 1$, where $\beta_k^\pm \in \mathbb{C}$ and $m_\pm \in \mathbb{N}$ are given numbers and where the functions $\mathbf{h}_k^\pm$ are defined by

$$\mathbf{h}_k^\pm(s) = \frac{(\mp 1)^k}{\pi \mathbf{i}} \frac{s^{k-1}}{(1+s)^k}\,, \quad s > 0\,,\ k \in \mathbb{N}\,.$$

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We assume that the coefficient functions a and b belong to the set $\mathbf{PC}$ of piecewise continuous functions[1] and that the right-hand side function f belongs to the weighted $\mathbf{L}^2$-space $\mathbf{L}^2_\nu$. The function $u \in \mathbf{L}^2_\nu$ stands for the unknown solution. The inner product and the norm in the Hilbert space $\mathbf{L}^2_\nu$ are given by

$$\langle u, v\rangle_\nu := \int_{-1}^{1} u(y)\overline{v(y)}\nu(y)\, dy \quad \text{and} \quad \|u\|_\nu := \sqrt{\langle u, u\rangle_\nu}\,,$$

respectively, where $\nu(x) = \sqrt{\frac{1+x}{1-x}}$ is the Chebyshev weight of third kind. Let

$$\mathcal{S} : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \quad u \mapsto \frac{1}{\pi \mathbf{i}} \int_{-1}^{1} \frac{u(y)\, dy}{y - \cdot}$$

be the Cauchy singular integral operator, $a\mathcal{I} : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu$, $u \mapsto au$ be the operator of multiplication by a and $\mathcal{B}^\pm_k : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$ with

$$\left(\mathcal{B}^\pm_k u\right)(x) = \int_{-1}^{1} \mathbf{h}^\pm_k \left(\frac{1 \mp x}{1 \mp y}\right) \frac{u(y)\, dy}{1 \mp y} = \frac{1}{\pi \mathbf{i}} \int_{-1}^{1} \frac{(1 \mp x)^{k-1} u(y)\, dy}{(y + x \mp 2)^k}$$

be the integral operators with a fixed singularity at ± 1. Then we can write (2.1) in the form

$$\mathcal{A}u := \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta^-_k \mathcal{B}^-_k + \sum_{k=1}^{m_+} \beta^+_k \mathcal{B}^+_k\right) u = f. \tag{2.2}$$

It is a well known fact that the single operators $a\mathcal{I}$, $\mathcal{S}$, and $\mathcal{B}^\pm_k$ are bounded in $\mathbf{L}^2_\nu$ (see [4, Theorem 1.16, Remark 8.3, and Theorem 9.1]). That means that these operators belong to the Banach algebra $\mathcal{L}(\mathbf{L}^2_\nu)$ of all bounded and linear operators $\mathcal{A} : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu$. In order to get approximate solutions of the integral equation we propose a polynomial collocation method in combination with a quadrature discretization of the integral operators with fixed singularities. For that, we need some further notations. Let $\sigma(x) = \frac{1}{\sqrt{1-x^2}}$, $\varphi(x) = \sqrt{1-x^2}$, and $\mu(x) = \sqrt{\frac{1-x}{1+x}}$ be the Chebyshev weights of first, second, and fourth kind, respectively. For $n \geq 0$ and $\tau \in \{\sigma, \varphi, \nu, \mu\}$, we denote by $p^\tau_n(x)$ the corresponding normalized Chebyshev polynomials of degree n w.r.t. the weight $\tau(x)$ and with positive leading coefficient, which we abbreviate by $T_n(x) = p^\sigma_n(x)$, $U_n(x) = p^\varphi_n(x)$,

[1] We call a function $a : [-1, 1] \to \mathbb{C}$ piecewise continuous if it is continuous at ± 1, if the one-sided limits $a(x \pm 0)$ exist for all $x \in (-1, 1)$ and at least one of them coincides with $a(x)$.

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$R_n(x) = p_n^\nu(x)$, and $P_n(x) = p_n^\mu(x)$. We know that

$$T_0(x) = \frac{1}{\sqrt{\pi}}, \quad T_n(\cos s) = \sqrt{\frac{2}{\pi}} \cos ns, \quad n \geq 1,\ s \in (0,\pi),$$

and, for $n \geq 0$, $s \in (0,\pi)$,

$$U_n(\cos s) = \frac{\sqrt{2}\sin(n+1)s}{\sqrt{\pi}\sin s}, \quad R_n(\cos s) = \frac{\cos(n+\frac{1}{2})s}{\sqrt{\pi}\cos\frac{s}{2}},$$

as well as

$$P_n(\cos s) = \frac{\sin(n+\frac{1}{2})s}{\sqrt{\pi}\sin\frac{s}{2}}.$$

The zeros x_{jn}^τ of $p_n^\tau(x)$ are given by

$$x_{jn}^\sigma = \cos\frac{j-\frac{1}{2}}{n}\pi, \quad x_{jn}^\varphi = \cos\frac{j\pi}{n+1}, \quad x_{jn}^\nu = \cos\frac{j-\frac{1}{2}}{n+\frac{1}{2}}\pi, \quad x_{jn}^\mu = \cos\frac{j\pi}{n+\frac{1}{2}},$$

$j = 1, \cdots, n$. We introduce the Lagrange interpolation operator $\mathcal{L}_n^\tau$ defined for every function $f : (-1,1) \to \mathbb{C}$ by

$$\mathcal{L}_n^\tau f = \sum_{j=1}^n f(x_{jn}^\tau)\ell_{jn}^\tau, \quad \ell_{jn}^\tau(x) = \frac{p_n^\tau(x)}{(x - x_{jn}^\tau)(p_n^\tau)'(x_{jn}^\tau)} = \prod_{k=1,k\neq j}^n \frac{x - x_{kn}^\tau}{x_{jn}^\tau - x_{kn}^\tau}\,.$$

The respective Christoffel numbers $\lambda_{jn}^\tau = \int\limits_{-1}^1 \ell_{jn}^\tau(x)\tau(x)\,dx$ are equal to

$$\lambda_{jn}^\sigma = \frac{\pi}{n}\,, \quad \lambda_{jn}^\varphi = \frac{\pi\left[1 - (x_{jn}^\varphi)^2\right]}{n+1}\,, \quad \lambda_{jn}^\nu = \frac{\pi(1 + x_{jn}^\nu)}{n+\frac{1}{2}}\,, \quad \lambda_{jn}^\mu = \frac{\pi(1 - x_{jn}^\mu)}{n+\frac{1}{2}}\,.$$

The collocation method seeks an approximation $u_n \in \mathbf{L}_\nu^2$ of the form

$$u_n(x) = \mu(x)p_n(x), \quad p_n \in \mathbf{P}_n \tag{2.3}$$

for a solution of equation (2.2) by solving

$$(\mathcal{A}u_n)(x_{jn}^\tau) = f(x_{jn}^\tau), \quad j = 1,2,\ldots,n, \tag{2.4}$$

where $\mathbf{P}_n$ denotes the set of all algebraic polynomials of degree less than $n \in \mathbb{N}$. Of course, here we assume, that the values $f(x)$ are well defined. Otherwise, we can replace f by an appropriate approximation f_n, cf. the remark on the convergence of the method at the end of this section. We set

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$$\widetilde{p}_n(x) := \mu(x)P_n(x), \quad n = 0, 1, 2, \ldots$$

Using the Lagrange basis $\widetilde{\ell}^\tau_{kn}(x) = \dfrac{\mu(x)\ell^\tau_{kn}(x)}{\mu(x^\tau_{kn})}$, $k = 1, \ldots, n$, in $\mu\mathbf{P}_n$, we can write u_n as

$$u_n = \sum_{j=0}^{n-1} \alpha_{jn}\widetilde{p}_j = \sum_{k=1}^{n} \xi_{kn}\widetilde{\ell}^\tau_{kn}.$$

If we introduce the Fourier projections

$$\mathcal{L}_n : \mathbf{L}^2_\nu \to \mathbf{L}^2_\nu, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_j\rangle_\nu \, \widetilde{p}_j$$

and the weighted interpolation operators $\mathcal{M}^\tau_n := \mu\mathcal{L}^\tau_n\mu^{-1}\mathcal{I}$, then the collocation system (2.4) can be written as the operator equation

$$\mathcal{M}^\tau_n\mathcal{A}\mathcal{L}_n u_n = \mathcal{M}^\tau_n f, \quad u_n \in \operatorname{im}\mathcal{L}_n. \tag{2.5}$$

Now we describe the above mentioned method based on an additional quadrature discretization. For $n \in \mathbb{N}$, $\tau \in \{\sigma, \mu\}$, and $k_0 \in \mathbb{N}$, on the space $\mathbf{C}(-1, 1)$ of continuous functions $u : (-1, 1) \longrightarrow \mathbb{C}$, we define the quadrature operators

$$\mathcal{H}^{\pm,\tau}_{n,k_0} : \mathbf{C}(-1, 1) \longrightarrow \mathbf{C}[-1, 1],$$

$$\begin{aligned} u \mapsto (\omega^\tau_n)^2 \sum_{k=1}^{n} \varphi(x^\tau_{kn})\mathbf{h}^\pm_{k_0}\left(\frac{1 \mp x}{1 \mp x^\tau_{kn}}\right)\frac{u(x^\tau_{kn})}{1 \mp x^\tau_{kn}} \\ = \sum_{k=1}^{n} \lambda^\tau_{kn}\mathbf{h}^\pm_{k_0}\left(\frac{1 \mp x}{1 \mp x^\tau_{kn}}\right)\frac{(\tau^{-1}u)(x^\tau_{kn})}{1 \mp x^\tau_{kn}}, \end{aligned} \tag{2.6}$$

where $\omega^\sigma_n = \sqrt{\frac{\pi}{n}}$ and $\omega^\mu_n = \sqrt{\frac{\pi}{n+\frac{1}{2}}}$. Instead of equation (2.5) we consider

$$\mathcal{A}^\tau_n u_n := \mathcal{M}^\tau_n\left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-}\beta^-_k\mathcal{H}^{-,\tau}_{n,k} + \sum_{k=1}^{m_+}\beta^+_k\mathcal{H}^{+,\tau}_{n,k}\right)\mathcal{L}_n u_n = \mathcal{M}^\tau_n f, \tag{2.7}$$

$u_n \in \operatorname{im}\mathcal{L}_n$. We call (2.7) a collocation–quadrature method. For studying the convergence of the approximate solution u_n to a solution u of the equation $\mathcal{A}u = f$, we have to investigate the stability of the collocation–quadrature method. We call a sequence (or a method) $(\mathcal{A}_n)_{n=1}^\infty =: (\mathcal{A}_n)$ of linear operators $\mathcal{A}_n : \operatorname{im}\mathcal{L}_n \longrightarrow \operatorname{im}\mathcal{L}_n$ stable, if the operators $\mathcal{A}_n$ are invertible for all sufficiently large $n \in \mathbb{N}$ and if the norms $\left\|\mathcal{A}_n^{-1}\mathcal{L}_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}$ are uniformly bounded. If the method is stable, then the strong convergence of $\mathcal{A}_n\mathcal{L}_n$ to $\mathcal{A} \in \mathcal{L}(\mathbf{L}^2_\nu)$ implies the injectivity of the operator $\mathcal{A}$ and, if additionally the image of $\mathcal{A}$

equals $\mathbf{L}^2_\nu$, then the convergence $f_n \longrightarrow f$, $f_n \in \operatorname{im}\mathcal{L}_n$, in $\mathbf{L}^2_\nu$ implies the $\mathbf{L}^2_\nu$-convergence of the solution $u_n \in \operatorname{im}\mathcal{L}_n$ of $\mathcal{A}_n u_n = f_n$ to the (unique) solution $u \in \mathbf{L}^2_\nu$ of $\mathcal{A}u = f$. The stability of the collocation method $(\mathcal{M}^\tau_n \mathcal{A}\mathcal{L}_n)$ was investigated in [1]. In the present paper we focus on the collocation–quadrature method $(\mathcal{A}^\tau_n)$ defined in (2.7).

3. The C^*-algebra framework for the stability of operator sequences

In what follows we will consider the operator sequence under consideration as an element of a C^*-algebra. For the definition of this algebra, we need some spaces as well as some useful operators. By ℓ^2 we denote the Hilbert space of all square summable sequences $\xi = (\xi_j)_{j=0}^\infty$, $\xi_j \in \mathbb{C}$ with the inner product $\langle \xi, \eta\rangle = \sum_{j=0}^\infty \xi_j \overline{\eta_j}$. Additionally, we define the following operators

$$\mathcal{W}_n : \mathbf{L}^2_\nu \longrightarrow \mathbf{L}^2_\nu\,, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_{n-1-j}\rangle_\nu \widetilde{p}_j\,,$$

$$\mathcal{P}_n : \ell^2 \longrightarrow \ell^2\,, \quad (\xi_j)_{j=0}^\infty \mapsto (\xi_0, \cdots, \xi_{n-1}, 0, \ldots)\,,$$

and, for $\tau \in \{\sigma, \mu\}$ and $\chi^\tau_{kn} = \sqrt{1 + x^\tau_{kn}}$,

$$\mathcal{V}^\tau_n : \operatorname{im}\mathcal{L}_n \longrightarrow \operatorname{im}\mathcal{P}_n, \quad u \mapsto (\omega^\tau_n \chi^\tau_{1n} u(x^\tau_{1n}), \ldots, \omega^\tau_n \chi^\tau_{nn} u(x^\tau_{nn}), 0, \ldots)\,,$$

$$\widetilde{\mathcal{V}}^\tau_n : \operatorname{im}\mathcal{L}_n \longrightarrow \operatorname{im}\mathcal{P}_n, \quad u \mapsto (\omega^\tau_n \chi^\tau_{nn} u(x^\tau_{nn}), \ldots, \omega^\tau_n \chi^\tau_{1n} u(x^\tau_{1n}), 0, \ldots)\,.$$

Let $T = \{1, 2, 3, 4\}$, set

$$\mathbf{X}^{(1)} = \mathbf{X}^{(2)} = \mathbf{L}^2_\nu,\ \mathbf{X}^{(3)} = \mathbf{X}^{(4)} = \ell^2,\ \mathcal{L}^{(1)}_n = \mathcal{L}^{(2)}_n = \mathcal{L}_n,\ \mathcal{L}^{(3)}_n = \mathcal{L}^{(4)}_n = \mathcal{P}_n,$$

and define $\mathcal{E}^{(t)}_n : \operatorname{im}\mathcal{L}_n \longrightarrow \mathbf{X}^{(t)}_n := \operatorname{im}\mathcal{L}^{(t)}_n$ for $t \in T$ by

$$\mathcal{E}^{(1)}_n = \mathcal{L}_n, \quad \mathcal{E}^{(2)}_n = \mathcal{W}_n, \quad \mathcal{E}^{(3)}_n = \mathcal{V}^\tau_n, \quad \mathcal{E}^{(4)}_n = \widetilde{\mathcal{V}}^\tau_n.$$

Here and at other places, we use the notion $\mathcal{L}_n$, $\mathcal{W}_n$, ... instead of $\mathcal{L}_n|_{\operatorname{im}\mathcal{L}_n}$, $\mathcal{W}_n|_{\operatorname{im}\mathcal{L}_n}$, ..., respectively. All operators $\mathcal{E}^{(t)}_n$, $t \in T$, are invertible with the inverses

$$\left(\mathcal{E}^{(1)}_n\right)^{-1} = \mathcal{E}^{(1)}_n,\ \left(\mathcal{E}^{(2)}_n\right)^{-1} = \mathcal{E}^{(2)}_n,\ \left(\mathcal{E}^{(3)}_n\right)^{-1} = (\mathcal{V}^\tau_n)^{-1},\ \left(\mathcal{E}^{(4)}_n\right)^{-1} = (\widetilde{\mathcal{V}}^\tau_n)^{-1},$$

where, for $\xi \in \operatorname{im}\mathcal{P}_n$,

$$(\mathcal{V}^\tau_n)^{-1}\xi = (\omega^\tau_n)^{-1} \sum_{k=1}^n \frac{\xi_{k-1}}{\sqrt{1 + x^\tau_{kn}}}\, \widetilde{\ell}^\tau_{kn}$$

and

$$\left(\widetilde{V}_n^{\tau}\right)^{-1}\xi = (\omega_n^{\tau})^{-1}\sum_{k=1}^{n}\frac{\xi_{n-k}}{\sqrt{1+x_{kn}^{\tau}}}\,\widetilde{\ell}_{kn}^{\tau}.$$

Now we can introduce the algebra of operator sequences we are interested in. By $\mathfrak{F} = \mathfrak{F}^{\tau}$ we denote the set of all sequences $(\mathcal{A}_n)$ of linear operators $\mathcal{A}_n : \operatorname{im}\mathcal{L}_n \longrightarrow \operatorname{im}\mathcal{L}_n$ for which the strong limits

$$\mathcal{W}^t(\mathcal{A}_n) := \lim_{n\to\infty}\mathcal{E}_n^{(t)}\mathcal{A}_n\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}$$

and

$$\left(\mathcal{W}^t(\mathcal{A}_n)\right)^* = \lim_{n\to\infty}\left(\mathcal{E}_n^{(t)}\mathcal{A}_n\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}\right)^*$$

for all $t \in T$ exist. If $\mathfrak{F}$ is provided with the supremum norm

$$\|(\mathcal{A}_n)\|_{\mathfrak{F}} := \sup_{n\geq 1}\|\mathcal{A}_n\mathcal{L}_n\|_{\mathfrak{L}(\mathbf{L}^2_\nu)}$$

and with the algebraic operations $(\mathcal{A}_n) + (\mathcal{B}_n) := (\mathcal{A}_n + \mathcal{B}_n)$, $(\mathcal{A}_n)(\mathcal{B}_n) := (\mathcal{A}_n\mathcal{B}_n)$, and $(\mathcal{A}_n)^* := (\mathcal{A}_n^*)$, one can easily check that $\mathfrak{F}$ becomes a C^*-algebra with the identity element $(\mathcal{L}_n)$. Moreover, we introduce the set $\mathfrak{J} \subset \mathfrak{F}$ of all sequences of the form

$$\left(\sum_{t=1}^{4}\left(\mathcal{E}_n^{(t)}\right)^{-1}\mathcal{L}_n^{(t)}\mathcal{T}_t\mathcal{E}_n^{(t)} + \mathcal{C}_n\right),$$

where the linear operators $\mathcal{T}_t : \mathbf{X}^{(t)} \longrightarrow \mathbf{X}^{(t)}$ are compact and where the sequence $(\mathcal{C}_n) \in \mathfrak{F}$ belongs to the closed ideal $\mathfrak{G}$ of all sequences from $\mathfrak{F}$ tending to zero in norm, i.e., $\lim\limits_{n\to\infty}\|\mathcal{C}_n\mathcal{L}_n\|_{\mathfrak{L}(\mathbf{L}^2_\nu)} = 0$. From [1, Section 2] and [2,3, Theorem 10.33] we get the following proposition.

Proposition 3.1. *The set $\mathfrak{J}$ forms a two-sided closed ideal in the C^*-algebra $\mathfrak{F}$. Moreover, a sequence $(\mathcal{A}_n) \in \mathfrak{F}$ is stable if and only if the operators $\mathcal{W}^t(\mathcal{A}_n) : \mathbf{X}^{(t)} \to \mathbf{X}^{(t)}$, $t \in T$ and the coset $(\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J}$ are invertible.*

4. The limit operators

For $\alpha, \beta \geq 0$, by $\mathbf{C}_{\alpha,\beta} = \mathbf{C}_{\alpha,\beta}(-1,1)$ we denote the set of all continuous functions $f : (-1,1) \to \mathbb{C}$, for which the finite limits

$$c_{\pm} := \lim_{x\to\pm 1}(1-x)^{\alpha}(1+x)^{\beta}f(x)$$

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exist and where $c_+ = 0$ if $\alpha > 0$ and $c_- = 0$ if $\beta > 0$. Equipped with the norm

$$\|f\|_{\alpha,\beta,\infty} = \sup\left\{(1-x)^\alpha(1+x)^\beta|f(x)| : -1 < x < 1\right\}$$

$\mathbf{C}_{\alpha,\beta}$ becomes a Banach space. In case $\alpha = \beta = 0$, we write $(\mathbf{C}, \|.\|_\infty)$ instead of $\left(\mathbf{C}_{0,0}, \|.\|_{0,0,\infty}\right)$.

In the sequel, by c we will denote some positive real constant which can assume different values at different places.

Lemma 4.1. *(See [5], Theorem 9.25.) Let $\widetilde{\mu}, \widetilde{\nu}$ be classical Jacobi weights with $\widetilde{\mu}$, $\widetilde{\mu}\widetilde{\nu} \in \mathbf{L}^1(-1,1)$ and let $j \in \mathbb{N}$ be fixed. Then, for each polynomial q with $\deg q \le jn$,*

$$\sum_{k=1}^n \lambda_{kn}^{\widetilde{\mu}} |q(x_{kn}^{\widetilde{\mu}})| \widetilde{\nu}(x_{kn}^{\widetilde{\mu}}) \le c \int_{-1}^1 |q(x)|\widetilde{\mu}(x)\widetilde{\nu}(x)\,dx,$$

where the constant c does not depend on n and q and where

$$x_{kn}^{\widetilde{\mu}} \quad \textit{and} \quad \lambda_{kn}^{\widetilde{\mu}} = \int_{-1}^1 \ell_{kn}^{\widetilde{\mu}}(x)\widetilde{\mu}(x)\,dx$$

are the nodes and the Christoffel numbers of the Gaussian rule with respect to the weight $\widetilde{\mu}$.

Lemma 4.2. *Let $\vartheta(x) = (1-x)^\gamma(1+x)^\delta$ with $\gamma, \delta > -1$.*

(a) *If $0 \le \alpha < \gamma + 1$ and $0 \le \beta < \delta + 1$, then*

$$\lim_{n\to\infty} \sum_{k=1}^n \lambda_{kn}^\vartheta f(x_{kn}^\vartheta) = \int_{-1}^1 f(x)\vartheta(x)\,dx \quad \forall\, f \in \mathbf{C}_{\alpha,\beta}\,.$$

(b) *If $0 \le \alpha < \frac{\gamma+1}{2}$ and $0 \le \beta < \frac{\delta+1}{2}$, then*

$$\lim_{n\to\infty} \left\|f - \mathcal{L}_n^\vartheta f\right\|_\vartheta = 0 \quad \forall\, f \in \mathbf{C}_{\alpha,\beta}\,.$$

Proof. Define the linear functionals $F_n : \mathbf{C}_{\alpha,\beta} \longrightarrow \mathbb{C}$ by $F_n(f) = \sum_{k=1}^n \lambda_{kn}^\vartheta f(x_{kn}^\vartheta)$. Then, due to Lemma 4.1 (set $q(x) \equiv 1$, $\widetilde{\nu}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}$, $\widetilde{\mu}(x) = \vartheta(x)$)

$$|F_n(f)| \le \sum_{k=1}^n \frac{\lambda_{kn}^\vartheta}{(1-x_{kn}^\vartheta)^\alpha(1+x_{kn}^\vartheta)^\beta}\, \|f\|_{\alpha,\beta,\infty} \le c\,\|f\|_{\alpha,\beta,\infty}\,,$$

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where the constant does not depend on n and f. Hence, the linear functionals F_n are uniformly bounded. Moreover,

$$\lim_{n\to\infty} F_n(f) = \int_{-1}^{1} f(x)\vartheta(x)\,dx$$

for all algebraic polynomials $f(x)$, and the Banach–Steinhaus theorem gives assertion (a), since the set of algebraic polynomials is dense in $\mathbf{C}_{\alpha,\beta}$. To prove (b), we mention that, taking into account the algebraic accuracy of the Gaussian rule and again Lemma 4.1,

$$\left\|\mathcal{L}_n^\vartheta f\right\|_\vartheta^2 = \sum_{k=1}^n \lambda_{kn}^\vartheta |f(x_{kn}^\vartheta)|^2 \le c\,\|f\|_{\alpha,\beta,\infty}^2 \quad \forall\, f \in \mathbf{C}_{\alpha,\beta}\,.$$

Thus, the operators $\mathcal{L}_n^\vartheta : \mathbf{C}_{\alpha,\beta} \longrightarrow \mathbf{L}_\vartheta^2$ are uniformly bounded. Since the relation $\lim\limits_{n\to\infty} \left\|f - \mathcal{L}_n^\vartheta f\right\|_\vartheta^2 = 0$ holds for all algebraic polynomials $f(x)$, the assertion again follows by the Banach–Steinhaus theorem. □

Corollary 4.3. *Let $0 \le \alpha < \frac{1}{4}$, $0 \le \beta < \frac{3}{4}$, and $f \in \mathbf{C}_{\alpha,\beta}$. Then, for $\tau = \sigma$ and $\tau = \mu$, $\mathcal{M}_n^\tau f \longrightarrow f$ in $\mathbf{L}_\nu^2$.*

Proof. In case $\tau = \mu$, we have $\|\mathcal{M}_n^\mu f - f\|_\nu = \left\|\mathcal{L}_n^\mu \mu^{-1} f - \mu^{-1} f\right\|_\mu$ and $\mu^{-1} f \in \mathbf{C}_{\max\{\alpha,\varepsilon\}+\frac{1}{2},\max\{0,\beta-\frac{1}{2}\}}$ for all $\varepsilon > 0$. Lemma 4.2(b) gives the assertion.

Now, let $\tau = \sigma$ and $\varepsilon > 0$. Since $\mu^{-1} f$ belongs to $\mathbf{L}_\mu^2$, there is a polynomial $p(x)$ such that $\|f - \mu p\|_\nu = \left\|\mu^{-1} f - p\right\|_\mu < \frac{\varepsilon}{2}$. For $n > \deg p$, we have

$$\begin{aligned} \|\mathcal{M}_n^\sigma f - f\|_\nu &\le \|\mathcal{M}_n^\sigma (f - \mu p)\|_\nu + \|\mu p - f\|_\nu \\ &= \left\|\mathcal{L}_n^\sigma (\mu^{-1} f - p)\right\|_\mu + \|\mu p - f\|_\nu < \left\|\mathcal{L}_n^\sigma (\mu^{-1} f - p)\right\|_\mu + \frac{\varepsilon}{2} \end{aligned}$$

and, using the Gaussian rule w.r.t. the weight σ,

$$\begin{aligned} \left\|\mathcal{L}_n^\sigma (\mu^{-1} f - p)\right\|_\mu^2 &= \int_{-1}^{1} (1-x)\left|\left(\mathcal{L}_n^\sigma (\mu^{-1} f - p)\right)(x)\right|^2 \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{\pi}{n} \sum_{k=1}^n (1 - x_{kn}^\sigma) \left| \sqrt{\frac{1 + x_{kn}^\sigma}{1 - x_{kn}^\sigma}}\, f(x_{kn}^\sigma) - p(x_{kn}^\sigma) \right|^2 \\ &= \frac{\pi}{n} \sum_{k=1}^n \left| \sqrt{1 + x_{kn}^\sigma}\, f(x_{kn}^\sigma) - \sqrt{1 - x_{kn}^\sigma}\, p(x_{kn}^\sigma) \right|^2 , \end{aligned}$$

where $\sqrt{1+\cdot}\, f - \sqrt{1-\cdot}\, p \in \mathbf{C}_{\alpha,\max\{0,\beta-\frac{1}{2}\}}$. Lemma 4.2(a) implies the inequality $\limsup\limits_{n\to\infty} \|\mathcal{M}_n^\sigma f - f\|_\nu < \varepsilon$. This proves the corollary. □

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Let, for $k_0 \in \mathbb{N}$, $\widetilde{\mathcal{B}}_{k_0}^{\pm} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ with

$$\left(\widetilde{\mathcal{B}}_{k_0}^{\pm} u\right)(x) = \int_{-1}^{1} \mathbf{h}_{k_0}^{\pm}\left(\frac{1 \mp y}{1 \mp x}\right) \frac{u(y)\,dy}{1 \mp x} = \frac{1}{\pi \mathbf{i}} \int_{-1}^{1} \frac{(1 \mp y)^{k_0-1} u(y)\,dy}{(y + x \mp 2)^{k_0}}$$

being integral operators with fixed singularities at ± 1. In case $\tau(x) = v^{\alpha,\beta}(x) := (1-x)^\alpha \cdot (1+x)^\beta$, we also write $\mathbf{L}^2_{\alpha,\beta} = \left(\mathbf{L}^2_{\alpha,\beta}, \|.\|_{\alpha,\beta}\right)$ instead of $\mathbf{L}^2_\tau = \mathbf{L}^2_{v^{\alpha,\beta}} = \left(\mathbf{L}^2_{v^{\alpha,\beta}}, \|.\|_{v^{\alpha,\beta}}\right)$.

Lemma 4.4. *Let $\alpha, \beta \geq 0$, $-1 < \gamma, \delta < 1$, and $k_0 \in \mathbb{N}$.*

(a) *If $\frac{1+\delta}{2} < \beta$, then $\mathcal{B}_{k_0}^-, \widetilde{\mathcal{B}}_{k_0}^- \in \mathcal{L}\left(\mathbf{L}^2_{\gamma,\delta}, \mathbf{C}_{0,\beta}\right)$.*
(b) *If $\frac{1+\gamma}{2} < \alpha$, then $\mathcal{B}_{k_0}^+, \widetilde{\mathcal{B}}_{k_0}^+ \in \mathcal{L}\left(\mathbf{L}^2_{\gamma,\delta}, \mathbf{C}_{\alpha,0}\right)$.*

Proof. To prove (a) we mention that, for $u \in \mathbf{L}^2_{\gamma,\delta}$, the functions $\mathcal{B}_{k_0}^- u, \widetilde{\mathcal{B}}_{k_0}^- u : (-1, 1] \to \mathbb{C}$ are continuous (cf. [1, after (7.10)]). Choose an ε such that $0 < \varepsilon < \beta - \frac{1+\delta}{2}$. Then, for $u \in \mathbf{L}^2_{\gamma,\delta}$,

$$\begin{aligned}\left|\left(v^{0,\beta}\mathcal{B}_{k_0}^- u\right)(x)\right| &= \frac{v^{0,\beta}(x)}{\pi}\left|\int_{-1}^{1} \frac{(1+x)^{k_0-1} u(y)\,dy}{(2+x+y)^{k_0}}\right| \leq \frac{v^{0,\beta}(x)}{\pi}\int_{-1}^{1} \frac{|u(y)|\,dy}{2+x+y} \\ &\leq \frac{v^{0,\beta}(x)}{\pi}\sqrt{\int_{-1}^{1} \frac{v^{-\gamma,-\delta}(y)\,dy}{(2+x+y)^2}}\;\|u\|_{\gamma,\delta} \\ &\leq \frac{v^{0,\varepsilon}(x)}{\pi}\sqrt{\int_{-1}^{1} \frac{v^{-\gamma,-\delta}(y)\,dy}{(1+y)^{2-2\beta+2\varepsilon}}}\;\|u\|_{\gamma,\delta} = c\, v^{0,\varepsilon}(x)\,\|u\|_{\gamma,\delta}\,.\end{aligned}$$

In the same way one can prove the assertion for $\widetilde{\mathcal{B}}_{k_0}^-$. The proof of (b) is completely analogous to the proof of (a). □

Remark 4.5. The results of Lemma 4.4 can be modified, for example in the following way. For $\alpha, \beta \geq 0$, let $\mathbf{BC}_{\alpha,\beta}$ denote the Banach space of all continuous functions $f : (-1, 1) \longrightarrow \mathbb{C}$, for which $v^{\alpha,\beta} f : (-1, 1) \longrightarrow \mathbb{C}$ is a bounded function, with the norm $\|f\|_{\alpha,\beta,\infty} := \sup\left\{v^{\alpha,\beta}(x)|f(x)| : -1 < x < 1\right\}$. If

$$-1 < \gamma, \delta < 1 \quad \text{and} \quad \beta = \frac{1+\delta}{2}\,,$$

then $\mathcal{B}_{k_0}^-, \widetilde{\mathcal{B}}_{k_0}^- \in \mathcal{L}\left(\mathbf{L}^2_{\gamma,\delta}, \mathbf{BC}_{0,\beta}\right)$. Indeed, as in the proof of Lemma 4.4 we have

$$\left|\left(v^{0,\beta}\mathcal{B}_{k_0}^- u\right)(x)\right| \le \frac{(1+x)^\beta}{\pi}\sqrt{\int_{-1}^{1}\frac{v^{-\gamma,-\delta}(y)\,dy}{(2+x+y)^2}}\,\|u\|_{\gamma,\delta}$$

and, moreover,

$$\begin{aligned}\int_{-1}^{1}\frac{v^{-\gamma,-\delta}(y)\,dy}{(2+x+y)^2} &\le c\left(\int_{-1}^{0}\frac{dy}{(2+x+y)^2(1+y)^\delta}+1\right)\\ &\le c\left(\frac{1}{(1+x)^{1+\delta}}\int_0^\infty\frac{dv}{(1+v)^2v^\delta}+1\right),\end{aligned}$$

giving $\left\|\mathcal{B}_{k_0}^- u\right\|_{0,\beta,\infty} \le c\,\|u\|_{\gamma,\delta}$. Analogously, $\mathcal{B}_{k_0}^+, \widetilde{\mathcal{B}}_{k_0}^+ \in \mathcal{L}\left(\mathbf{L}^2_{\gamma,\delta}, \mathbf{BC}_{a,0}\right)$ for $\alpha = \frac{1+\gamma}{2}$.

If $A(x,n,\ldots)$ and $B(x,n,\ldots)$ are two positive functions depending on certain variables $x,n,\ldots$, then we will write $A \sim_{x,n,\ldots} B$, if there is a positive constant c not depending on $x,n,\ldots$, such that

$$c^{-1}B(x,n,\ldots) \le A(x,n,\ldots) \le c\,B(x,n,\ldots)$$

holds.

Lemma 4.6. *(See [1], Lemma 7.4.) Let $\tau \in \{\sigma,\varphi,\nu,\mu\}$ and $k = 0,1,\ldots,n \in \mathbb{N}$. For the zeros x_{kn}^τ of the orthogonal polynomials p_n^τ, we have, for all sufficiently large n,*

$$\int_{x_{k+1,n}^\tau}^{x_{kn}^\tau}\sqrt{\frac{1+x}{1-x}}\,dx \sim_{k,n} \frac{1}{n}(1+x_{kn}^\tau),$$

where $x_{0n}^\tau := 1,\ x_{n+1,n}^\tau := -1$.

Corollary 4.7. *Using the relations $-x_{kn}^\sigma = x_{n+1-k,n}^\sigma$, $-x_{kn}^\varphi = x_{n+1-k,n}^\varphi$, $-x_{kn}^\nu = x_{n+1-k,n}^\mu$, $-x_{kn}^\mu = x_{n+1-k,n}^\nu$, $k = 0,1,\ldots,n+1$, and the substitution $y = -x$, one gets for all sufficiently large n,*

$$\int_{x_{k+1,n}^\tau}^{x_{kn}^\tau}\sqrt{\frac{1-x}{1+x}}\,dx \sim_{k,n} \frac{1}{n}(1-x_{k+1,n}^\tau),\quad \tau \in \{\sigma,\varphi,\nu,\mu\}\,,\ k = 0,1,\ldots,n\,.$$

Lemma 4.8. *For $\tau \in \{\sigma, \mu\}$ and $k_0 \in \mathbb{N}$, $\mathcal{M}_n^\tau \mu \widetilde{\mathcal{B}}_{k_0}^\pm \nu \mathcal{L}_n \longrightarrow \mu \widetilde{\mathcal{B}}_{k_0}^\pm \nu \mathcal{I}$ in the sense of strong convergence in $\mathbf{L}_\nu^2$.*

Proof. Let $u \in \mathbf{L}_\nu^2$. With the help of the exactness of the Gaussian rule for polynomials of degree less than $2n$, we have

$$\|\mathcal{M}_n^\sigma \mu \widetilde{\mathcal{B}}_k^\pm \nu u\|_\nu^2 = \|\sqrt{1-\cdot}\, \mathcal{L}_n^\sigma \widetilde{\mathcal{B}}_k^\pm \nu u\|_\sigma^2 = \frac{\pi}{n} \sum_{k=1}^n (1 - x_{kn}^\sigma) |(\widetilde{\mathcal{B}}_k^\pm \nu u)(x_{kn}^\sigma)|^2$$

and

$$\|\mathcal{M}_n^\mu \mu \widetilde{\mathcal{B}}_k^\pm \nu u\|_\nu^2 = \|\mathcal{L}_n^\mu \widetilde{\mathcal{B}}_k^\pm \nu u\|_\mu^2 = \frac{\pi}{n+\frac{1}{2}} \sum_{k=1}^n (1 - x_{kn}^\mu) |(\widetilde{\mathcal{B}}_k^\pm \nu u)(x_{kn}^\mu)|^2 .$$

Taking into account Corollary 4.7 we can estimate, for $\tau = \sigma$, $\tau = \mu$, and for all sufficiently large n,

$$\begin{aligned}
\|\mathcal{M}_n^\tau \mu \widetilde{\mathcal{B}}_k^+ \nu u\|_\nu^2 &\le c \sum_{k=1}^n \int_{x_{kn}^\tau}^{x_{k-1,n}^\tau} \sqrt{\frac{1-x}{1+x}}\, dx \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1-y)^{k-1} \nu(y) |u(y)|\, dy}{(2 - y - x_{kn}^\tau)^k} \right)^2 \\
&\le c \sum_{k=1}^n \int_{x_{kn}^\tau}^{x_{k-1,n}^\tau} \sqrt{\frac{1-x}{1+x}} \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1-y)^{k-1} \nu(y) |u(y)|\, dy}{(2 - y - x)^k} \right)^2 dx \\
&\le c \int_{-1}^1 \mu(x) \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1-y)^{k-1} \nu(y) |u(y)|\, dy}{(2 - y - x)^k} \right)^2 dx \\
&= c \left\| \widetilde{\mathcal{B}}_k^+ \nu |u| \right\|_\mu^2 \le c \left\| \widetilde{\mathcal{B}}_k^+ \right\|_{\mathcal{L}(\mathbf{L}_\mu^2)}^2 \|u\|_\nu^2 .
\end{aligned}$$

To handle $\|\mathcal{M}_n^\tau \mu \widetilde{\mathcal{B}}_k^- \nu u\|_\nu$, we use the estimate

$$\int_{x_{k+1,n}^\tau}^{x_{kn}^\tau} \sqrt{\frac{1-x}{1+x}}\, dx \ge \frac{\pi}{2n} \left(1 - x_{kn}^\tau\right), \quad k = 1, \ldots, n,\ \tau \in \{\sigma, \mu\} ,$$

and get

$$\begin{aligned}
\|\mathcal{M}_n^\tau \mu \widetilde{\mathcal{B}}_k^- \nu u\|_\nu^2 &\le 2 \sum_{k=1}^n \int_{x_{k+1,n}^\tau}^{x_{kn}^\tau} \sqrt{\frac{1-x}{1+x}}\, dx \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1+y)^{k-1} |\nu(y) u(y)|\, dy}{(2 + y + x_{kn}^\tau)^k} \right)^2 \\
&\le 2 \sum_{k=1}^n \int_{x_{k+1,n}^\tau}^{x_{kn}^\tau} \sqrt{\frac{1-x}{1+x}} \left(\frac{1}{\pi} \int_{-1}^1 \frac{(1+y)^{k-1} \nu(y) |u(y)|\, dy}{(2 + y + x)^k} \right)^2 dx
\end{aligned}$$

$$\leq 2\int_{-1}^{1}\mu(x)\left(\frac{1}{\pi}\int_{-1}^{1}\frac{(1+y)^{k-1}\nu(y)|u(y)|\,dy}{(2+y+x)^k}\right)^2 dx$$
$$= 2\left\|\widetilde{\mathcal{B}}_k^-\nu|u|\,\right\|_\mu^2 \leq 2\left\|\widetilde{\mathcal{B}}_k^-\right\|_{\mathcal{L}(\mathbf{L}_\mu^2)}^2 \|u\|_\nu^2 .$$

Since the operators $\widetilde{\mathcal{B}}_k^\pm : \mathbf{L}_\mu^2 \longrightarrow \mathbf{L}_\mu^2$ are bounded (cf. [4, Remark 8.3 and Theorem 9.1]), we get the uniform boundedness of the sequences $\left(\mathcal{M}_n^\tau \mu\widetilde{\mathcal{B}}_k^\pm \nu\mathcal{I}\right)$. If $u : [-1,1] \longrightarrow \mathbb{C}$ is a continuous function with compact support in $(-1,1)$, then $\left(\widetilde{\mathcal{B}}_k^\pm \nu u\right)(x)$ is continuous on $[-1,1]$ and, consequently, $\mu\widetilde{\mathcal{B}}_k\nu u \in \mathbf{C}_{0,\frac{1}{2}+\varepsilon}$ for $\varepsilon > 0$. By Corollary 4.3 and the Banach–Steinhaus theorem we get the strong convergence $\mathcal{M}_n^\tau\mu\widetilde{\mathcal{B}}_k^\pm\nu\mathcal{I} \longrightarrow \mu\widetilde{\mathcal{B}}_k^\pm\nu\mathcal{I}$ in $\mathbf{L}_\nu^2$. $\square$

Lemma 4.9. *Let $\tau \in \{\sigma,\mu\}$ and $k_0 \in \mathbb{N}$. Then*

$$\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n \longrightarrow \mathcal{B}_{k_0}^\pm \quad \text{and} \quad (\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n)^* \longrightarrow -\mu\widetilde{\mathcal{B}}_{k_0}^\pm\nu\mathcal{I}$$

in the sense of strong convergence in $\mathbf{L}_\nu^2$.

Proof. First of all we show the uniform boundedness of the operator sequences under consideration. We have (see [1, (7.15), (7.16)])

$$\begin{aligned}
&\mathcal{E}_n^{(3)}\mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n(\mathcal{E}_n^{(3)})^{-1}\mathcal{L}_n^{(3)} \\
&\quad = \left[\frac{(\omega_n^\tau)^2}{\pi\mathbf{i}}\,\frac{\sqrt{1-x_{kn}^\tau}\sqrt{1+x_{jn}^\tau}(1\mp x_{jn}^\tau)^{k_0-1}}{(x_{kn}^\tau+x_{jn}^\tau\mp 2)^{k_0}}\right]_{j,k=1}^n \\
&\quad = \mathcal{E}_n^{(3)}\mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{L}_n(\mathcal{E}_n^{(3)})^{-1}\mathcal{L}_n^{(3)} + \mathcal{K}_n^{\tau,\pm},
\end{aligned} \tag{4.1}$$

where the matrix operators $\mathcal{K}_n^{\tau,\pm} = \left[\,a_{jk}^{\tau,\pm}\,\right]_{j,k=1}^n$ are given by

$$\begin{aligned}
a_{jk}^{\sigma,\pm} &= \sqrt{\frac{\pi}{2}}\,\frac{(-1)^{k+1}}{n}\,\frac{\sqrt{1+x_{kn}^\sigma}}{x_{kn}^\sigma+x_{jn}^\sigma\mp 2}\,\cdot \\
&\quad\cdot\sum_{t=1}^{k_0}\frac{(1\mp x_{jn}^\sigma)^{k_0-t}}{(x_{kn}^\sigma+x_{jn}^\sigma\mp 2)^{k_0-t}}\sqrt{1+x_{jn}^\sigma}(-1)^{t-1}\widetilde{h}_n^t(-x_{jn}^\sigma\pm 2)
\end{aligned}$$

and

$$\begin{aligned}
a_{jk}^{\mu,\pm} &= \sqrt{\frac{\pi}{2}}\,\frac{(-1)^{k+1}}{n+\frac{1}{2}}\,\frac{\varphi(x_{kn}^\mu)}{x_{kn}^\mu+x_{jn}^\mu\mp 2}\,\cdot \\
&\quad\cdot\sum_{t=1}^{k_0}\frac{(1\mp x_{jn}^\mu)^{k_0-t}}{(x_{kn}^\mu+x_{jn}^\mu\mp 2)^{k_0-t}}\sqrt{1+x_{jn}^\mu}\,(-1)^{t-1}h_n^t(-x_{jn}^\mu\pm 2)
\end{aligned}$$

with $\widetilde{h}_n^k$ and h_n^k defined in [1, (7.6)–(7.8)]. One can show that, for $j = 1, \dots, n$ (see [1, (7.18)–(7.21)])

$$\left| \sqrt{1 + x_{jn}^\tau}\, h_n^t(-x_{jn}^\tau - 2) \right| \le c(n+1-j)^{t-1} e^{-n+j} \le c, \tag{4.2}$$

$$\left| \sqrt{1 + x_{jn}^\tau}\, \widetilde{h}_n^t(-x_{jn}^\tau - 2) \right| \le c(n+1-j)^{t-1} e^{-n+j} \le c, \tag{4.3}$$

$$\left| \frac{\widetilde{h}_n^t(-x_{jn}^\tau + 2)}{\sqrt{1 - x_{jn}^\tau}} \right| \le c\, j^{t-1} e^{-j} \le c,$$

and

$$\left| h_n^t(-x_{jn}^\tau + 2) \right| \le c\, j^{t-1} e^{-j} \le c, \quad j = 1, \dots, n.$$

Since the operators $\mathcal{M}_n^\tau \mathcal{B}_{k_0}^\pm \mathcal{L}_n : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ are uniformly bounded [1, Lemma 7.5] and since $\mathcal{E}_n^{(3)} : \operatorname{im} \mathcal{L}_n \longrightarrow \operatorname{im} \mathcal{P}_n$ is a unitary operator [6, p. 733], it suffices to show that the operators $\mathcal{K}_n^{\tau,\pm} : \operatorname{im} \mathcal{P}_n \longrightarrow \operatorname{im} \mathcal{P}_n$ are uniformly bounded. For $\mathcal{K}_n^{\sigma,-}$, this follows from

$$\begin{aligned}
\sum_{k=1}^n \sum_{j=1}^n |a_{jk}^{\sigma,-}|^2 &\le c \sum_{k=1}^n \sum_{j=1}^n \left[\frac{1}{n} \frac{\sqrt{1 + x_{kn}^\sigma}}{x_{kn}^\sigma + x_{jn}^\sigma + 2} \sum_{t=1}^{k_0} \sqrt{1 + x_{jn}^\sigma} \left| \widetilde{h}_n^t(-x_{jn}^\sigma - 2) \right| \right]^2 \\
&\le c \sum_{k=1}^n \sum_{j=1}^n \left[\frac{1}{n\sqrt{1 + x_{kn}^\sigma}} (n+1-j)^{k_0-1} e^{-n+j} \right]^2 \qquad (4.4) \\
&\le c \sum_{k=1}^n \frac{1}{(n+1-k)^2} \sum_{j=1}^n (n+1-j)^{2k_0-2} e^{-2(n-j)} \\
&= c \sum_{k=1}^n \frac{1}{k^2} \sum_{j=1}^n j^{2k_0-2} e^{-2(j-1)} \le c\,,
\end{aligned}$$

where we used relation (4.3) and the relation

$$\sqrt{1 + x_{kn}^\tau} \sim_{k,n} \frac{n+1-k}{n}\,, \quad k = 1, \dots, n\,,\ n \in \mathbb{N}\,, \tag{4.5}$$

which follows from $\frac{\pi}{2} - s \ge \cos s \ge 1 - \frac{2s}{\pi}$, $0 \le s \le \frac{\pi}{2}$. The uniform boundedness of $\mathcal{K}_n^{\sigma,+}$ and $\mathcal{K}_n^{\mu,\pm}$ can be shown in the same way.

It remains to prove the convergence on a dense subset of $\mathbf{L}_\nu^2$. For this, we take the set of all functions of the form $v(x) = \mu(x)(1+x)^2 p(x)$, where $p(x)$ is a polynomial. Using Corollary 4.3, for $n > \deg p + 2$, we get

$$
\begin{aligned}
&\left\|\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{-,\tau} \mathcal{L}_n v - \mathcal{B}_{k_0}^- v\right\|_\nu \\
&\quad \le \|\mathcal{M}_n^\tau\|_{\mathcal{L}(\mathbf{C},\mathbf{L}_\nu^2)} \left\|\mathcal{H}_{n,k_0}^{-,\tau} v - \mathcal{B}_{k_0}^- v\right\|_\infty + \left\|\mathcal{M}_n^\tau \mathcal{B}_{k_0}^- v - \mathcal{B}_{k_0}^- v\right\|_\nu \\
&\quad \le c \left\|\mathcal{H}_{n,k_0}^{-,\tau} v - \mathcal{B}_{k_0}^- v\right\|_\infty + \left\|\mathcal{M}_n^\tau \mathcal{B}_{k_0}^- v - \mathcal{B}_{k_0}^- v\right\|_\nu ,
\end{aligned}
$$

where $\lim\limits_{n\to\infty} \left\|\mathcal{M}_n^\tau \mathcal{B}_{k_0}^- v - \mathcal{B}_{k_0}^- v\right\|_\nu = 0$ (see [1, Lemma 7.5]). If $R_n(f)_\tau$ denotes the quadrature error

$$
R_n(f)_\tau = \int_{-1}^{1} f(x)\tau(x)\,dx - \sum_{k=1}^{n} \lambda_{kn}^\tau f(x_{kn}^\tau)\,,
$$

then [7, (5.1.35)]

$$
|R_n(f)_\tau| \le \frac{c}{n} \int_{-1}^{1} |f'(x)|\tau(x)\varphi(x)\,dx\,, \tag{4.6}
$$

where the constant does not depend on n and f. Let $\tau = \sigma$. Then

$$
\begin{aligned}
&\left(\mathcal{H}_{n,k_0}^{-,\sigma} v\right)(x) - \left(\mathcal{B}_{k_0}^- v\right)(x) \\
&= \frac{1}{\pi \mathbf{i}} \left[\frac{\pi}{n} \sum_{k=1}^{n} \frac{(1+x)^{k_0-1}\varphi(x_{kn}^\sigma)v(x_{kn}^\sigma)}{(2+x+x_{kn}^\sigma)^{k_0}} - \int_{-1}^{1} \frac{(1+x)^{k_0-1}v(y)\,dy}{(2+x+y)^{k_0}} \right] \\
&= \frac{1}{\pi \mathbf{i}} R_n(f_x)_\sigma
\end{aligned}
$$

with $f_x(y) = \dfrac{(1+x)^{k_0-1}v_0(y)}{(2+x+y)^{k_0}}$ and $v_0(y) = \varphi(y)v(y) = (1+y)^2(1-y)p(y)$. We estimate

$$
\begin{aligned}
\int_{-1}^{1} |f_x'(y)|\,dy &\le \int_{-1}^{1} \frac{(1+x)^{k_0-1}|v_0'(y)|\,dy}{(2+x+y)^{k_0}} + k_0 \int_{-1}^{1} \frac{(1+x)^{k_0-1}|v_0(y)|\,dy}{(2+x+y)^{k_0+1}} \\
&\le c\left(\|p'\|_\infty + \|p\|_\infty\right),
\end{aligned}
$$

which implies, using (4.6), $\lim\limits_{n\to\infty} \left\|\mathcal{H}_{n,k_0}^{-,\tau} v - \mathcal{B}_{k_0}^- v\right\|_\infty = 0$. If $\tau = \mu$, then

$$
\left(\mathcal{H}_{n,k_0}^{-,\mu} v\right)(x) - \left(\mathcal{B}_{k_0}^- v\right)(x) = \frac{1}{\pi \mathbf{i}} R_n(f_x)_\mu
$$

with $f_x(y) = \dfrac{(1+x)^{k_0-1}v_0(y)}{(2+x+y)^{k_0}}$ and $v_0(y) = \nu(y)v(y) = (1+y)^2 p(y)$. The integral

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$$\int_{-1}^{1} |f_x'(y)|\mu(y)\varphi(y)\,dy = \int_{-1}^{1} |f_x'(y)|(1-y)\,dy$$

can be estimated analogously. The convergence of $\mathcal{H}_{n,k_0}^{+,\tau}$ on a dense subset of $\mathbf{L}_\nu^2$ can be proved in the same way.

Now we compute the adjoint operators. Using the abbreviation $h(x,y) := \mathbf{h}_{k_0}^{\pm}\left(\frac{1\mp x}{1\mp y}\right)\cdot\frac{1}{1\mp y}$ and the relation $\frac{1}{\nu(x)\tau(x)} = \begin{cases} 1-x & : \ \tau=\sigma, \\ 1 & : \ \tau=\mu, \end{cases}$ for functions $u, v \in \mathbf{L}_\nu^2$ we have

$$\begin{aligned}
&\left\langle \mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n u, v\right\rangle_\nu \\
&= \left\langle \mathcal{L}_n^\tau \nu \mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n u, (\nu\tau)^{-1}\nu\mathcal{L}_n v\right\rangle_\tau \\
&= \sum_{j=1}^n \lambda_{jn}^\tau \nu(x_{jn}^\tau)\left(\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n u\right)(x_{jn}^\tau)\,\overline{(\tau^{-1}\mathcal{L}_n v)(x_{jn}^\tau)} \\
&= \sum_{j=1}^n \lambda_{jn}^\tau \nu(x_{jn}^\tau)\sum_{k=1}^n \lambda_{kn}^\tau h(x_{jn}^\tau, x_{kn}^\tau)\left(\tau^{-1}\mathcal{L}_n u\right)(x_{kn}^\tau)\,\overline{(\tau^{-1}\mathcal{L}_n v)(x_{jn}^\tau)} \\
&= \sum_{k=1}^n \lambda_{kn}^\tau \nu(x_{kn}^\tau)\left(\tau^{-1}\mathcal{L}_n u\right)(x_{kn}^\tau)\;\cdot \\
&\qquad \cdot\,\overline{\mu(x_{kn}^\tau)\sum_{j=1}^n \lambda_{jn}^\tau \overline{h(x_{jn}^\tau, x_{kn}^\tau)}\nu(x_{jn}^\tau)\,(\tau^{-1}\mathcal{L}_n v)(x_{jn}^\tau)} \\
&= \left\langle u, -\mathcal{M}_n^\tau \mu \widetilde{\mathcal{H}}_{n,k_0}^{\pm,\tau}\nu\mathcal{L}_n v\right\rangle_\nu,
\end{aligned}$$

where

$$\begin{aligned}
\left(\widetilde{\mathcal{H}}_{n,k_0}^{\pm,\tau} f\right)(x) &= (\omega_n^\tau)^2 \sum_{j=1}^n \varphi(x_{jn}^\tau)\mathbf{h}_{k_0}^{\pm}\left(\frac{1\mp x_{jn}^\tau}{1\mp x}\right)\frac{f(x_{jn}^\tau)}{1\mp x} \\
&= \sum_{j=1}^n \lambda_{jn}^\tau \mathbf{h}_{k_0}^{\pm}\left(\frac{1\mp x_{jn}^\tau}{1\mp x}\right)\frac{(\tau^{-1}f)(x_{jn}^\tau)}{1\mp x}.
\end{aligned}$$

Consequently, $(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n)^* = -\mathcal{M}_n^\tau \mu \widetilde{\mathcal{H}}_{n,k_0}^{\pm,\tau}\nu\mathcal{L}_n$. Let

$$v(x) = \mu(x)(1+x)^2(1-x)p(x)$$

with a polynomial $p(x)$. Then

$$\begin{aligned}
&\left(\mu\widetilde{\mathcal{H}}_{n,k_0}^{-,\sigma}\nu v\right)(x) - \left(\mu\widetilde{\mathcal{B}}_{k_0}^{-}\nu v\right)(x) \\
&\quad = \frac{\mu(x)}{\pi\mathbf{i}}\Big[\frac{\pi}{n}\sum_{j=1}^n \frac{(1+x_{jn}^\sigma)^{k_0-1}\nu(x_{jn}^\sigma)\varphi(x_{jn}^\sigma)v(x_{jn}^\sigma)}{(2+x+x_{jn}^\sigma)^{k_0}}
\end{aligned}$$

$$- \int_{-1}^{1} \frac{(1+y)^{k_0-1}\nu(y)v(y)\,dy}{(2+x+y)^{k_0}} \Bigg]$$

$$= \frac{\mu(x)}{\pi\mathbf{i}} R_n(f_x)_\sigma \,,$$

where $f_x(y) = \dfrac{v_0(y)}{(2+x+y)^{k_0}}$ and

$$v_0(y) = (1+y)^{k_0-1}\nu(y)\varphi(y)v(y) = (1-y)^{\frac{3}{2}}(1+y)^{k_0+\frac{3}{2}}p(y)\,.$$

Relation (4.6) yields, for $\varepsilon > 0$,

$$\lim_{n\to\infty} \left\| \mu \widetilde{\mathcal{H}}_{n,k_0}^{-,\sigma} \nu v - \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu v \right\|_{0,\frac{1}{2}+\varepsilon,\infty} = 0\,.$$

Due to Corollary 4.3 and the strong convergence of $\mathcal{M}_n^\sigma \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu \mathcal{L}_n \longrightarrow \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu \mathcal{I}$ in $\mathbf{L}^2_\nu$ (see Lemma 4.8), we can conclude, for $0 < \varepsilon < \frac{1}{4}$,

$$\begin{aligned}
&\left\| \mathcal{M}_n^\sigma \mu \widetilde{\mathcal{H}}_{n,k_0}^{-,\sigma} \nu \mathcal{L}_n v - \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu v \right\|_\nu \\
&\quad \le \|\mathcal{M}_n^\sigma\|_{\mathcal{L}(\mathbf{C}_{0,\frac{1}{2}+\varepsilon}, \mathbf{L}^2_\nu)} \left\| \mu \widetilde{\mathcal{H}}_{n,k_0}^{-,\sigma} \nu v - \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu v \right\|_{0,\frac{1}{2}+\varepsilon,\infty} \\
&\qquad + \left\| \mathcal{M}_n^\sigma \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu v - \mu \widetilde{\mathcal{B}}_{k_0}^{-} \nu v \right\|_\nu \longrightarrow 0
\end{aligned}$$

if $n \longrightarrow \infty$, and the strong convergence of $(\mathcal{M}_n^\sigma \mathcal{H}_{n,k_0}^{-,\sigma} \mathcal{L}_n)^*$ is proved. Analogously one can handle the operators $(\mathcal{M}_n^\mu \mathcal{H}_{n,k_0}^{-,\mu} \mathcal{L}_n)^*$ and $(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{+,\tau} \mathcal{L}_n)^*$, $\tau = \sigma, \mu$. □

We recall the following version of Lebesgue's dominated convergence theorem.

Lemma 4.10. *If $\xi, \eta \in \ell^2$, $\xi^n = (\xi_k^n)_{k=1}^\infty$, $|\xi_k^n| \le |\eta_k|$ for all $k = 0, 1, 2, \ldots$ and for all $n \ge n_0$ and if $\lim_{n\to\infty} \xi_k^n = \xi_k$ for all $k = 0, 1, 2, \ldots$, then $\lim_{n\to\infty} \|\xi^n - \xi\|_{\ell^2} = 0$.*

Set

$$\mathbf{A}_{k_0} = \left[2\ \mathbf{h}_{k_0}^{-} \left(\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2} \right) \frac{j+\frac{1}{2}}{(k+\frac{1}{2})^2} \right]_{j,k=0}^{\infty},$$

$$\mathbf{A}_{k_0}^\sigma = \left[2\ \mathbf{h}_{k_0}^{+} \left(\frac{(j+\frac{1}{2})^2}{(k+\frac{1}{2})^2} \right) \frac{1}{k+\frac{1}{2}} \right]_{j,k=0}^{\infty},$$

and

$$\mathbf{A}_{k_0}^\mu = \left[2\ \mathbf{h}_{k_0}^{+} \left(\frac{(j+1)^2}{(k+1)^2} \right) \frac{1}{k+1} \right]_{j,k=0}^{\infty}.$$

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Lemma 4.11. *For $\tau \in \{\sigma, \mu\}$ and $k_0 \in \mathbb{N}$, the strong limits*

$$\mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{-,\tau} \mathcal{L}_n) = \Theta \quad \text{and} \quad \mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{+,\tau} \mathcal{L}_n) = \mathbf{A}_{k_0}^\tau$$

exist, where Θ is the zero operator in ℓ^2. Moreover, the respective sequences of the adjoint operators converge strongly.

Proof. We have to show the strong convergence of the sequence

$$\left(\mathcal{E}_n^{(3)} \mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n (\mathcal{E}_n^{(3)})^{-1} \mathcal{L}_n^{(3)} \right)$$

in ℓ^2. Since the uniform boundedness of $\left(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n \right)$ is already known (due to Lemma 4.9) and since the operators $\mathcal{E}_n^{(3)}$ are unitary, it suffices to check the convergence on the elements $e_m = (\delta_{j,m})_{j=0}^\infty$, $m = 0, 1, 2, \ldots$ of the standard basis of ℓ^2. For $n > m = k - 1 \geq 0$, we can write (cf. (4.1))

$$\begin{aligned} &\mathcal{E}_n^{(3)} \mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n (\mathcal{E}_n^{(3)})^{-1} \mathcal{L}_n^{(3)} e_m \\ &= \left[(\omega_n^\tau)^2 \sqrt{1 + x_{jn}^\tau} \sqrt{1 - x_{kn}^\tau} \mathbf{h}_{k_0}^\pm \left(\frac{1 \mp x_{jn}^\tau}{1 \mp x_{kn}^\tau} \right) \frac{1}{1 \mp x_{kn}^\tau} \right]_{j=1}^n \\ &=: \left[b_{jk}^{\pm,\tau} \right]_{j=1}^n , \end{aligned}$$

where $\left[\xi_{j-1} \right]_{j=1}^n \in \mathbb{C}^n$ is identified with $(\xi_0, \ldots, \xi_{n-1}, \ldots) \in \ell^2$. Let $j, k \geq 1$ be fixed. Due to the continuity of $\mathbf{h}_{k_0}^\pm : (0, \infty) \longrightarrow \mathbb{C}$ we get, for $n \longrightarrow \infty$,

$$\begin{aligned} b_{jk}^{+,\sigma} &= \frac{\pi}{n} \frac{cs \frac{2j-1}{4n}\pi}{\sin \frac{2k-1}{4n}\pi} \mathbf{h}_{k_0}^+ \left(\frac{\sin^2 \frac{2j-1}{4n}\pi}{\sin^2 \frac{2k-1}{4n}\pi} \right) \longrightarrow \mathbf{h}_{k_0}^+ \left(\frac{\left(j - \frac{1}{2}\right)^2}{\left(k - \frac{1}{2}\right)^2} \right) \frac{2}{k - \frac{1}{2}} , \\ b_{jk}^{+,\mu} &= \frac{\pi}{n + \frac{1}{2}} \frac{cs \frac{j\pi}{2n+1}}{\sin \frac{k\pi}{2n+1}} \mathbf{h}_{k_0}^+ \left(\frac{\sin^2 \frac{j\pi}{2n+1}}{\sin^2 \frac{k\pi}{2n+1}} \right) \longrightarrow \mathbf{h}_{k_0}^+ \left(\frac{j^2}{k^2} \right) \frac{2}{k} , \\ b_{jk}^{-,\sigma} &= \frac{\pi}{n} \frac{cs \frac{2j-1}{4n}\pi \sin \frac{2k-1}{4n}\pi}{cs^2 \frac{2k-1}{4n}\pi} \mathbf{h}_{k_0}^+ \left(\frac{cs^2 \frac{2j-1}{4n}\pi}{cs^2 \frac{2k-1}{4n}\pi} \right) \longrightarrow 0 , \\ b_{jk}^{-,\mu} &= \frac{\pi}{n + \frac{1}{2}} \frac{cs \frac{j\pi}{2n+1} \sin \frac{k\pi}{2n+1}}{cs^2 \frac{k\pi}{2n+1}} \mathbf{h}_{k_0}^+ \left(\frac{cs^2 \frac{j\pi}{2n+1}}{cs^2 \frac{k\pi}{2n+1}} \right) \longrightarrow 0 . \end{aligned}$$

Let $k \geq 1$ be fixed, $n > k$, and $j = 1, \ldots, n$. Since, for $x > 0$, $\left| \mathbf{h}_{k_0}^\pm (x) \right| \leq (\pi x)^{-1}$, we get

$$\left| b_{jk}^{+,\sigma} \right| \leq \frac{1}{n} \frac{cs \frac{2j-1}{4n}\pi \sin \frac{2k-1}{4n}\pi}{\sin^2 \frac{2j-1}{4n}\pi} \leq c \, \frac{k}{j^2} ,$$

$$\left|b_{jk}^{+,\mu}\right| \le \frac{1}{n+\frac{1}{2}} \, \frac{cs\frac{j\pi}{2n+1}\,\sin\frac{k\pi}{2n+1}}{\sin^2\frac{j\pi}{2n+1}} \le c\,\frac{k}{j^2}\,,$$

$$\left|b_{jk}^{-,\sigma}\right| \le \frac{1}{n} \, \frac{\sin\frac{2k-1}{4n}\pi}{cs\frac{2j-1}{4n}\pi} \le c\,\frac{k}{n(n+1-j)} \le c\,\frac{k}{j}\,,$$

$$\left|b_{jk}^{-,\mu}\right| \le \frac{1}{n+\frac{1}{2}} \, \frac{\sin\frac{k\pi}{2n+1}}{cs\frac{j\pi}{2n+1}} \le c\,\frac{k}{n(n+1-j)} \le c\,\frac{k}{j}\,,$$

where the constants do not depend on j, k, n. Now, for fixed $k = m + 1$, Lemma 4.10 gives

$$\mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{+,\tau} \mathcal{L}_n) e_m = \mathbf{A}_{k_0}^\tau e_m \quad \text{and} \quad \mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{-,\tau} \mathcal{L}_n) e_m = \Theta e_m\,.$$

The convergence of the sequences of the adjoint operators can be seen analogously. □

In the same manner one can prove the following lemma.

Lemma 4.12. *For $\tau \in \{\sigma, \mu\}$ and $k_0 \in \mathbb{N}$, the strong limits*

$$\mathcal{W}^4(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{-,\tau} \mathcal{L}_n) = \mathbf{A}_{k_0} \quad \text{and} \quad \mathcal{W}^4(\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{+,\tau} \mathcal{L}_n) = \Theta$$

exist, where Θ is the zero operator in ℓ^2. Moreover, the respective sequences of the adjoint operators converge strongly.

In the following lemma we consider the difference between collocation and collocation–quadrature for the Mellin operators. The assertions will be used later for proving the existence of the second limit operator.

Lemma 4.13. *Let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$. Then, for $\tau \in \{\sigma, \mu\}$, there hold*

$$\left(\mathcal{M}_n^\tau (1 \mp \cdot)^\varepsilon \mathcal{L}_n \left[\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n - \mathcal{M}_n^\tau \mathcal{B}_{k_0}^{\pm} \mathcal{L}_n\right]\right) \in \mathfrak{G}$$

and

$$\left(\left[\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n - \mathcal{M}_n^\tau \mathcal{B}_{k_0}^{\pm} \mathcal{L}_n\right] \mathcal{M}_n^\tau (1 \mp \cdot)^\varepsilon \mathcal{L}_n\right) \in \mathfrak{G}\,.$$

Proof. By equation (4.1) we get

$$\begin{aligned} &\mathcal{M}_n^\tau (1 \mp \cdot)^\varepsilon \mathcal{L}_n \left[\mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n - \mathcal{M}_n^\tau \mathcal{B}_{k_0}^{\pm} \mathcal{L}_n\right] \\ &\quad = \mathcal{M}_n^\tau (1 \mp \cdot)^\varepsilon \mathcal{L}_n (\mathcal{E}_n^{(3)})^{-1} \mathcal{K}_n^{\tau,\pm} \mathcal{E}_n^{(3)} = (\mathcal{E}_n^{(3)})^{-1} \mathcal{D}_n^{\tau,\pm} \mathcal{K}_n^{\tau,\pm} \mathcal{E}_n^{(3)}, \end{aligned}$$

where $\mathcal{D}_n^{\tau,\pm} = \operatorname{diag}\left[\,(1 \mp x_{jn}^\tau)^\varepsilon\,\right]_{j=1}^n$ and, since $\mathcal{E}_n^{(3)}$ is unitary,

$$\left\|(\mathcal{E}_n^{(3)})^{-1}\mathcal{D}_n^{\tau,\pm}\mathcal{K}_n^{\tau,\pm}\mathcal{E}_n^{(3)}\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} = \left\|\mathcal{D}_n^{\tau,\pm}\mathcal{K}_n^{\tau,\pm}\right\|_{\mathcal{L}(\ell^2)} \le \left\|\mathcal{D}_n^{\tau,\pm}\mathcal{K}_n^{\tau,\pm}\right\|_F ,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Using (4.2), (4.3), and (4.5) (see also (4.4)), we get

$$\begin{aligned}\left\|\mathcal{D}_n^{\tau,-}\mathcal{K}_n^{\tau,-}\right\|_F^2 &\le c\sum_{k=1}^n\sum_{j=1}^n\left[\frac{(1+x_{jn}^\tau)^\varepsilon}{n\sqrt{1+x_{kn}^\tau}}\,(n+1-j)^{k_0-1}e^{-n+j}\right]^2\\ &\le c\sum_{k=1}^n\frac{1}{(n+1-k)^2}\sum_{j=1}^n\left[\frac{(n+1-j)^{2\varepsilon}}{n^{2\varepsilon}}(n+1-j)^{k_0-1}e^{-n+j}\right]^2\\ &= \frac{c}{n^{4\varepsilon}}\sum_{k=1}^n\frac{1}{k^2}\sum_{j=1}^n\left[j^{k_0+2\varepsilon-1}e^{-(j-1)}\right]^2 \le \frac{c}{n^{4\varepsilon}}\,.\end{aligned}$$

Hence,

$$\left\|(\mathcal{E}_n^{(3)})^{-1}\mathcal{D}_n^{\tau,-}\mathcal{K}_n^{\tau,-}\mathcal{E}_n^{(3)}\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \longrightarrow 0, \quad n \longrightarrow \infty\,.$$

The same holds true for $(\mathcal{E}_n^{(3)})^{-1}\mathcal{D}_n^{\tau,+}\mathcal{K}_n^{\tau,+}\mathcal{E}_n^{(3)}$. The second assertion can be proved analogously. □

Lemma 4.14. *(See [1], Lemma 4.3.) Let $f \in \mathbf{C}[-1,1]$ and $k_0 \in \mathbb{N}$. Then, for $\tau \in \{\sigma,\mu\}$ there holds*

$$\left(\mathcal{M}_n^\tau f\mathcal{L}_n\mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{L}_n - \mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{L}_n\mathcal{M}_n^\tau f\mathcal{L}_n\right) \in \mathfrak{J}.$$

Lemma 4.15. *Let $f \in \mathbf{C}[-1,1]$ and $k_0 \in \mathbb{N}$. Then, for $\tau \in \{\sigma,\mu\}$, there holds*

$$\left(\mathcal{M}_n^\tau f\mathcal{L}_n\mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n - \mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n\mathcal{M}_n^\tau f\mathcal{L}_n\right) \in \mathfrak{J}. \tag{4.7}$$

Proof. Again by equation (4.1) we get

$$\begin{aligned}&\mathcal{M}_n^\tau f\mathcal{L}_n\mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n - \mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n\mathcal{M}_n^\tau f\mathcal{L}_n\\ &\quad= \mathcal{M}_n^\tau f\mathcal{L}_n\mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{L}_n - \mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{L}_n\mathcal{M}_n^\tau f\mathcal{L}_n + (\mathcal{E}_n^{(3)})^{-1}\left(\mathcal{F}_n^\tau\mathcal{K}_n^{\tau,\pm} - \mathcal{K}_n^{\tau,\pm}\mathcal{F}_n^\tau\right)\mathcal{E}_n^{(3)},\end{aligned}$$

where $\mathcal{F}_n^\tau = \operatorname{diag}\left[\,f(x_{jn}^\tau)\,\right]_{j=1}^n$. In view of the estimate [6, (3.11)]

$$\left\|\mathcal{M}_n^\tau f_1\mathcal{L}_n - \mathcal{M}_n^\tau f_2\mathcal{L}_n\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)} \le c\left\|f_1 - f_2\right\|_\infty$$

and the closedness of $\mathfrak{J}$ as well as Lemma 4.14, it is sufficient to verify

$$\left((\mathcal{E}_n^{(3)})^{-1}\left(\mathcal{F}_n^\tau\mathcal{K}_n^{\tau,\pm}-\mathcal{K}_n^{\tau,\pm}\mathcal{F}_n^\tau\right)\mathcal{E}_n^{(3)}\right)\in\mathfrak{J}$$

for Lipschitz continuous functions f. In that case, by (4.2), (4.3), and (4.5) it follows

$$\begin{aligned}
&\left\|\mathcal{F}_n^\tau\mathcal{K}_n^{\tau,-}-\mathcal{K}_n^{\tau,-}\mathcal{F}_n^\tau\right\|_F^2\\
&\le c\sum_{k=1}^n\sum_{j=1}^n\left[\frac{1}{n}\frac{\sqrt{1+x_{kn}^\tau}\,[f(x_{jn}^\tau)-f(x_{kn}^\tau)]}{x_{kn}^\tau+x_{jn}^\tau+2}(n+1-j)^{k_0-1}e^{-n+j}\right]^2\\
&\le c\sum_{k=1}^n\sum_{j=1}^n\left[\frac{1}{n}\frac{\sqrt{1+x_{kn}^\tau}(x_{jn}^\tau-x_{kn}^\tau)}{x_{kn}^\tau+x_{jn}^\tau+2}(n+1-j)^{k_0-1}e^{-n+j}\right]^2\\
&\le\frac{c}{n^2}\sum_{k=1}^n\sum_{j=1}^n\left[\frac{(n+1-k)(j-k)}{(n+1-k)^2+(n+1-j)^2}\,(n+1-j)^{k_0-1}e^{-n+j}\right]^2\\
&=\frac{c}{n^2}\sum_{k=1}^n\sum_{j=1}^n\left[\frac{k(k-j)}{k^2+j^2}\,j^{k_0-1}e^{-(j-1)}\right]^2\\
&\le\frac{c}{n^2}\sum_{k=1}^n\sum_{j=1}^n j^{2k_0-2}e^{-2(j-1)}\le\frac{c}{n}\,.
\end{aligned}$$

Thus,

$$\left\|(\mathcal{E}_n^{(3)})^{-1}\left(\mathcal{F}_n^\tau\mathcal{K}_n^{\tau,-}-\mathcal{K}_n^{\tau,-}\mathcal{F}_n^\tau\right)\mathcal{E}_n^{(3)}\right\|_{\mathcal{L}(\mathbf{L}^2_\nu)}\longrightarrow 0,\quad n\longrightarrow\infty\,.$$

The case regarding the singularity at the point $+1$ can be dealt similarly. □

Lemma 4.16. *(See [1], Lemma 7.14.) Let $\tau\in\{\sigma,\mu\}$ and $k_0\in\mathbb{N}$. Then, we have $\mathcal{W}^2(\mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{L}_n)=\Theta$, where Θ is the zero operator in $\mathbf{L}^2_\nu$ and where the sequences of the adjoint operators $\left(\mathcal{W}_n\mathcal{M}_n^\tau\mathcal{B}_{k_0}^\pm\mathcal{W}_n\right)^*$ also converge strongly.*

Lemma 4.17. *If $\tau\in\{\sigma,\mu\}$ and $k_0\in\mathbb{N}$, then $\mathcal{W}^2(\mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{L}_n)=\Theta$ and the sequences of the adjoint operators $\left(\mathcal{W}_n\mathcal{M}_n^\tau\mathcal{H}_{n,k_0}^{\pm,\tau}\mathcal{W}_n\right)^*$ also converge strongly in $\mathbf{L}^2_\nu$.*

Proof. Let $f\in\mathbf{C}[-1,1]$ be vanishing in a neighborhood of ±1. The set of these functions is dense in $\mathbf{L}^2_\nu$. Set $\widetilde{f}_i=\mathcal{J}_i^{-1}f$, $i\in\{1,2\}$, where $\mathcal{J}_1,\mathcal{J}_2$ are the isometries defined in (4.8), (4.9) below. Choose a smooth cut-off function $\chi:[-1,1]\longrightarrow\mathbb{R}$, which vanishes in a neighborhood of ±1, such that $\chi f=f$ is fulfilled. If we define

$$\mathcal{L}_n^1:\mathbf{L}^2_\nu\longrightarrow\mathbf{L}^2_\nu\,,\quad u\mapsto\sum_{j=0}^{n-1}\langle u,R_j\rangle_\nu\,R_j\,,$$

then we have the strong convergence $\mathcal{L}_n^\sigma \chi \mathcal{L}_n^1 \longrightarrow \chi\mathcal{I}$ in $\mathbf{L}_\nu^2$. This follows from

$$\begin{aligned}&\left\|\mathcal{L}_n^\sigma \chi \mathcal{L}_n^1 u\right\|_\nu^2 = \left\|\sqrt{1+\cdot}\,\mathcal{L}_n^\sigma \chi \mathcal{L}_n^1 u\right\|_\sigma^2 \\ &\quad = \frac{\pi}{n}\sum_{k=1}^n (1+x_{kn}^\sigma)\left|\chi(x_{kn}^\sigma)\left(\mathcal{L}_n^1 u\right)(x_{kn}^\sigma)\right|^2 \le \|\chi\|_\infty^2 \left\|\mathcal{L}_n^1 u\right\|_\nu^2 \le \|\chi\|_\infty^2 \|u\|_\nu^2 \,,\end{aligned}$$

from $\|.\|_\nu \le c\,\|.\|_\sigma$, and from Lemma 4.2(b) using the Banach–Steinhaus theorem. Taking into account the relation (cf. [6, (3.19)])

$$\mathcal{W}_n \mathcal{M}_n^\sigma \chi \mathcal{L}_n \mathcal{W}_n = \mathcal{J}_1^{-1} \mathcal{L}_n^\sigma \chi \mathcal{J}_1 \mathcal{L}_n$$

together with $\mathcal{J}_1 \mathcal{L}_n \mathcal{J}_1^{-1} = \mathcal{L}_n^1$, we derive

$$\begin{aligned}&\left\|\mathcal{W}_n \mathcal{M}_n^\sigma \mathcal{H}_{n,k_0}^{\pm,\sigma} \mathcal{L}_n \mathcal{M}_n^\sigma \chi \mathcal{L}_n \mathcal{W}_n \widetilde{f}_1 - \mathcal{W}_n \mathcal{M}_n^\sigma \mathcal{H}_{n,k_0}^{\pm,\sigma} \mathcal{W}_n \widetilde{f}_1\right\|_\nu \\ &\quad \le c\left\|\mathcal{W}_n \mathcal{M}_n^\sigma \chi \mathcal{L}_n \mathcal{W}_n \widetilde{f}_1 - \widetilde{f}_1\right\|_\nu = c\left\|\mathcal{J}_1^{-1}\mathcal{L}_n^\sigma \chi \mathcal{J}_1 \mathcal{L}_n \mathcal{J}_1^{-1} f - \mathcal{J}_1^{-1} f\right\|_\nu \\ &\quad = c\left\|\mathcal{L}_n^\sigma \chi \mathcal{L}_n^1 f - f\right\|_\nu \longrightarrow 0.\end{aligned}$$

In case of the nodes of fourth kind, we use the formula (cf. [6, (3.20)])

$$\mathcal{W}_n \mathcal{M}_n^\mu \chi \mathcal{L}_n \mathcal{W}_n = \mathcal{J}_2^{-1} \sqrt{1-\cdot}\,\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}} \chi \mathcal{J}_2$$

and the relation $\mathcal{J}_2 \mathcal{L}_n \mathcal{J}_2^{-1} = \mathcal{L}_n^2$, where

$$\mathcal{L}_n^2 : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2 \,, \quad u \mapsto \sum_{j=0}^{n-1} \left\langle u, \sqrt{1-\cdot}\,U_j\right\rangle_\nu \sqrt{1-\cdot}\,U_j \,.$$

We get

$$\begin{aligned}&\left\|\mathcal{W}_n \mathcal{M}_n^\mu \mathcal{H}_{n,k_0}^{\pm,\mu} \mathcal{L}_n \mathcal{M}_n^\mu \chi \mathcal{L}_n \mathcal{W}_n \widetilde{f}_2 - \mathcal{W}_n \mathcal{M}_n^\mu \mathcal{H}_{n,k_0}^{\pm,\mu} \mathcal{L}_n \mathcal{W}_n \widetilde{f}_2\right\|_\nu \\ &\quad \le c\left\|\sqrt{1-\cdot}\,\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}} \chi \mathcal{L}_n^2 f - f\right\|_\nu \longrightarrow 0 \,,\end{aligned}$$

due to the strong convergence $\sqrt{1-\cdot}\,\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}} \chi \mathcal{L}_n^2 \longrightarrow \chi\mathcal{I}$ in $\mathbf{L}_\nu^2$. This strong convergence is a consequence of

$$\begin{aligned}\left\|\sqrt{1-\cdot}\,\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}}\, \chi \mathcal{L}_n^2 u\right\|_\nu^2 &= \left\|\sqrt{1+\cdot}\,\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}}\, \chi \mathcal{L}_n^2 u\right\|_\mu^2 \\ &= \frac{\pi}{n+\frac{1}{2}} \sum_{k=1}^n (1+x_{kn}^\mu)\left|\chi(x_{kn}^\mu)\left(\mathcal{L}_n^2 u\right)(x_{kn}^\mu)\right|^2\end{aligned}$$

$$\begin{aligned} &\le \|\chi\|_\infty^2 \left\| \sqrt{\frac{1+\cdot}{1-\cdot}}\,\mathcal{L}_n^2 u \right\|_\mu^2 \\ &= \|\chi\|_\infty^2 \left\|\mathcal{L}_n^2 u\right\|_\nu^2 \le \|\chi\|_\infty^2 \|u\|_\nu^2 \,, \end{aligned}$$

of (cf. Lemma 4.2)

$$\left\| \sqrt{1-\cdot}\,\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}}\,\chi u - \chi u \right\|_\nu = \left\| \sqrt{1+\cdot}\left[\mathcal{L}_n^\mu \frac{1}{\sqrt{1-\cdot}}\,\chi u - \frac{1}{\sqrt{1-\cdot}}\,\chi u \right] \right\|_\mu \longrightarrow 0$$

for $u \in \bigcup_{j=1}^\infty \operatorname{im} \mathcal{L}_j^2$, and of the Banach–Steinhaus theorem.

Consequently, it suffices to verify the strong convergence

$$\mathcal{W}_n \mathcal{M}_n^\tau \mathcal{H}_{n,k_0}^{\pm,\tau} \mathcal{L}_n \mathcal{M}_n^\tau \chi \mathcal{L}_n \mathcal{W}_n \longrightarrow \Theta \quad \text{in} \quad \mathbf{L}_\nu^2 .$$

But this follows from Lemma 4.13 and Lemma 4.16 taking into account that $\chi(x)(1\mp x)^{-\varepsilon}$ is a continuous function on $[-1,1]$. The convergence of the adjoint operators can be shown in the same manner. □

Define the isometries

$$\mathcal{J}_1 : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2, \qquad u \mapsto \sum_{j=0}^\infty \langle u, \widetilde{p}_j\rangle_\nu R_j, \tag{4.8}$$

$$\mathcal{J}_2 : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2, \qquad u \mapsto \sum_{j=0}^\infty \langle u, \widetilde{p}_j\rangle_\nu \sqrt{1-\cdot}\, U_j, \tag{4.9}$$

$$\mathcal{J}_3 : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2, \qquad u \mapsto \sum_{j=0}^\infty \langle u, \widetilde{p}_j\rangle_\nu \frac{1}{\sqrt{1+\cdot}}\, T_j$$

and the shift operator

$$\mathcal{V} : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2, \qquad u \mapsto \sum_{j=0}^\infty \langle u, \widetilde{p}_j\rangle_\nu \widetilde{p}_{j+1} .$$

The adjoint operators $\mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*, \mathcal{V}^* : \mathbf{L}_\nu^2 \to \mathbf{L}_\nu^2$ are given by

$$\mathcal{J}_1^* u = \mathcal{J}_1^{-1} u = \sum_{j=0}^\infty \langle u, R_j\rangle_\nu\, \widetilde{p}_j, \qquad \mathcal{J}_2^* u = \mathcal{J}_2^{-1} u = \sum_{j=0}^\infty \left\langle u, \sqrt{1-\cdot}\, U_j \right\rangle_\nu \widetilde{p}_j$$

and

$$\mathcal{J}_3^* u = \mathcal{J}_3^{-1} u = \sum_{j=0}^\infty \left\langle u, \frac{1}{\sqrt{1+\cdot}}\, T_j \right\rangle_\nu \widetilde{p}_j, \qquad \mathcal{V}^* u = \sum_{j=0}^\infty \langle u, \widetilde{p}_{j+1}\rangle_\nu\, \widetilde{p}_j .$$

Finally, we denote by $\mathbf{I} = [\delta_{jk}]_{j,k=0}^{\infty}$ the identity operator on ℓ^2 and by $\widetilde{\mathbf{S}}, \mathbf{S}^\tau : \ell^2 \longrightarrow \ell^2$ the operators defined by

$$\widetilde{\mathbf{S}} = \left[\frac{1-(-1)^{j-k}}{\pi\mathbf{i}(j-k)} + \frac{1-(-1)^{j+k+1}}{\pi\mathbf{i}(j+k+1)} \right]_{j,k=0}^{\infty} \tag{4.10}$$

and

$$\mathbf{S}^\tau = \begin{cases} \left[\dfrac{1-(-1)^{j-k}}{\pi\mathbf{i}(j-k)} - \dfrac{1-(-1)^{j+k+1}}{\pi\mathbf{i}(j+k+1)} \right]_{j,k=0}^{\infty} & : \quad \tau = \sigma, \\ \left[\dfrac{1-(-1)^{j-k}}{\pi\mathbf{i}} \left[\dfrac{1}{j-k} - \dfrac{1}{j+k+2} \right] \right]_{j,k=0}^{\infty} & : \quad \tau = \mu. \end{cases} \tag{4.11}$$

For $a, b \in \mathbf{PC}$, $m_\pm \in \mathbb{N}$ and $\beta_k^\pm \in \mathbb{C}$, we recall

$$\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2 \tag{4.12}$$

and

$$\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{H}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{H}_{n,k}^{+,\tau} \right) \mathcal{L}_n : \operatorname{im} \mathcal{L}_n \longrightarrow \operatorname{im} \mathcal{L}_n. \tag{4.13}$$

Moreover, we set (see the definitions of $\mathbf{A}_{k_0}$ and $\mathbf{A}_{k_0}^\tau$ after Lemma 4.10)

$$\mathbf{A} = \sum_{k_0=1}^{m_-} \beta_{k_0}^- \mathbf{A}_{k_0}, \quad \mathbf{A}^\sigma = \sum_{k_0=1}^{m_+} \beta_{k_0}^+ \mathbf{A}_{k_0}^\sigma, \quad \text{and } \mathbf{A}^\mu = \sum_{k_0=1}^{m_+} \beta_{k_0}^+ \mathbf{A}_{k_0}^\mu. \tag{4.14}$$

Proposition 4.18. *(See [1], Prop. 3.4.) Let $\tau \in \{\sigma, \mu\}$ and let $\mathcal{A}$ be defined by (4.12). The sequence $(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n)$ of the collocation method belongs to the algebra $\mathfrak{F}$ and the associated limit operators are given by*

$$\begin{aligned} \mathcal{W}^1(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n) &= \mathcal{A}, \\ \mathcal{W}^2(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n) &= \begin{cases} \mathcal{J}_1^{-1}(a\mathcal{J}_1 + \mathbf{i}b\mathcal{I}) & : \quad \tau = \sigma, \\ \mathcal{J}_2^{-1}(a\mathcal{J}_2 - \mathbf{i}b\mathcal{J}_3\mathcal{V}) & : \quad \tau = \mu, \end{cases} \\ \mathcal{W}^3(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n) &= a(1)\mathbf{I} + b(1)\mathbf{S}^\tau + \mathbf{A}^\tau + \mathbf{K}^\tau, \\ \mathcal{W}^4(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n) &= a(-1)\mathbf{I} - b(-1)\widetilde{\mathbf{S}} + \mathbf{A} + \mathbf{K}, \end{aligned}$$

where $\widetilde{\mathbf{S}}$, $\mathbf{S}$, $\mathbf{A}^\tau$, and $\mathbf{A}$ are defined in (4.10), (4.11) and (4.14) and where $\mathbf{K}$, $\mathbf{K}^\tau : \ell^2 \longrightarrow \ell^2$ are compact operators.

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Now, the following proposition is a consequence of Lemma 4.9, Lemma 4.11, Lemma 4.12, Lemma 4.17, and Proposition 4.18.

Proposition 4.19. *Let $\tau \in \{\sigma, \mu\}$. The sequence $(\mathcal{A}_n^\tau)$ of the collocation–quadrature method defined in (4.13) belongs to $\mathfrak{F}$ and the associated limit operators are given by*

$$\begin{aligned}
\mathcal{W}^1(\mathcal{A}_n^\tau) &= \mathcal{A}, \\
\mathcal{W}^2(\mathcal{A}_n^\tau) &= \begin{cases} \mathcal{J}_1^{-1}(a\mathcal{J}_1 + \mathbf{i}b\mathcal{I}): & \tau = \sigma, \\ \mathcal{J}_2^{-1}(a\mathcal{J}_2 - \mathbf{i}b\mathcal{J}_3\mathcal{V}): & \tau = \mu, \end{cases} \\
\mathcal{W}^3(\mathcal{A}_n^\tau) &= a(1)\mathbf{I} + b(1)\mathbf{S}^\tau + \mathbf{A}^\tau, \\
\mathcal{W}^4(\mathcal{A}_n^\tau) &= a(-1)\mathbf{I} - b(-1)\widetilde{\mathbf{S}} + \mathbf{A}.
\end{aligned}$$

5. The stability theorem for the collocation–quadrature methods

In this section we investigate the invertibility of the coset $(\mathcal{A}_n^\tau) + \mathfrak{J}$ in the quotient algebra $\mathfrak{F}/\mathfrak{J}$, where the sequence $(\mathcal{A}_n^\tau)$ of the collocation–quadrature method is given by (4.13). For that, we need some other operator sequences. Let $\mathbf{R}$ belong to the algebra $\operatorname{alg}\mathcal{T}(\mathbf{PC})$, which is the smallest C^*-subalgebra of the C^*-algebra $\mathcal{L}(\ell^2)$ of all linear and bounded operators in ℓ^2 containing all Toeplitz operators $\left[\,\widehat{g}_{j-k}\,\right]_{j,k=0}^{\infty}$ with piecewise continuous generating function $g(t) = \sum_{m\in\mathbb{Z}} \widehat{g}_m t^m$ defined on the unit circle $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$ and being continuous on $\mathbb{T} \setminus \{\pm 1\}$. We define the finite sections $\mathbf{R}_n := \mathcal{P}_n \mathbf{R} \mathcal{P}_n \in \mathcal{L}(\operatorname{im}\mathcal{P}_n)$ and set $\mathbf{R}_n^t := (\mathcal{E}_n^{(t)})^{-1}\mathbf{R}_n\mathcal{E}_n^{(t)}$, $t \in \{3, 4\}$.

Lemma 5.1. *(See [6], Lemma 5.4.) For $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ and $t \in \{3, 4\}$, the sequences $(\mathbf{R}_n^t)$ belong to the algebra $\mathfrak{F}$.*

Let $m_\pm \in \mathbb{N}$ be fixed. We denote by $\mathfrak{A} = \mathfrak{A}^\tau$ the smallest C^*-subalgebra of $\mathfrak{F}$ generated by all sequences of the ideal $\mathfrak{J}$, all sequences $(\mathbf{R}_n^t)$ with $t \in \{3, 4\}$ and $\mathbf{R} \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$, as well as by all sequences $(\mathcal{A}_n^\tau)$ with

$$\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{H}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{H}_{n,k}^{+,\tau} \right) \mathcal{L}_n\,, \quad a, b \in \mathbf{PC}\,,\ \beta_k^\pm \in \mathbb{C}\,.$$

Moreover, let $\mathfrak{A}_0 = \mathfrak{A}_0^\tau$ be the smallest C^*-subalgebra of $\mathfrak{F}$ containing all sequences from $\mathfrak{J}$ and all sequences $(\mathcal{A}_n^\tau)$ with

$$\mathcal{A}_n^\tau = \mathcal{M}_n^\tau \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{H}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{H}_{n,k}^{+,\tau} \right) \mathcal{L}_n\,, \quad a, b \in \mathbf{PC}\,,\ \beta_k^\pm \in \mathbb{C}\,.$$

For the coset $(\mathcal{A}_n) + \mathfrak{J}$ we use the abbreviation $(\mathcal{A}_n)^o$. Due to the property of inverse closedness of C^*-subalgebras, the invertibility of $(\mathcal{A}_n^\tau)^o$ in $\mathfrak{F}/\mathfrak{J}$ is equivalent to its invertibility in $\mathfrak{A}/\mathfrak{J}$. As a main tool for proving invertibility in the quotient algebra $\mathfrak{A}/\mathfrak{J}$ we will use the local principle of Allan and Douglas. For this, we have to find a C^*-subalgebra of the center of $\mathfrak{A}/\mathfrak{J}$ as well as its maximal ideal space. Lemma 4.15 together with [1, Lemma 4.3] shows that the set $\mathfrak{C} := \{(\mathcal{M}_n^\tau f \mathcal{L}_n)^o : f \in \mathbf{C}[-1,1]\}$ forms a $\mathcal{C}^*$ subalgebra of the center of $\mathfrak{A}/\mathfrak{J}$. This subalgebra is via the mapping $(\mathcal{M}_n^\tau f \mathcal{L}_n)^o \to f$ *-isomorphic to $\mathbf{C}[-1,1]$. The definition of this mapping is correct, since $(\mathcal{M}_n^\tau f_1 \mathcal{L}_n)^o = (\mathcal{M}_n^\tau f_2 \mathcal{L}_n)^o$ implies that $(f_1 - f_2)\mathcal{I} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ is compact, i.e., $f_1 - f_2 = 0$.

Consequently, the maximal ideal space of $\mathfrak{C}$ is equal to $\{\mathfrak{T}_\omega : \omega \in [-1,1]\}$ with

$$\mathfrak{T}_\omega := \{(\mathcal{M}_n^\tau f \mathcal{L}_n)^o : f \in \mathbf{C}[-1,1], f(\omega) = 0\}\,.$$

By $\mathfrak{J}_\omega = \mathfrak{J}_\omega^\tau$ we denote the smallest closed ideal of $\mathfrak{A}/\mathfrak{J}$, which contains $\mathfrak{T}_\omega$, i.e., $\mathfrak{J}_\omega$ is equal to the closure in $\mathfrak{A}/\mathfrak{J}$ of the set

$$\left\{ \sum_{j=1}^{m} \left(\mathcal{A}_n^j \mathcal{M}_n^\tau f_j \mathcal{L}_n\right)^o : \left(\mathcal{A}_n^j\right) \in \mathfrak{A},\ f_j \in \mathbf{C}[-1,1],\ f_j(\omega) = 0,\ m = 1, 2, \ldots \right\}.$$

The local principle of Allan and Douglas claims the following.

Proposition 5.2. *(Cf. [8], Sections 1.4.4, 1.4.6.) For all $\omega \in [-1,1]$, the ideal $\mathfrak{J}_\omega$ is a proper ideal in $\mathfrak{A}/\mathfrak{J}$. An element $(\mathcal{A}_n)^o$ of $\mathfrak{A}/\mathfrak{J}$ is invertible if and only if $(\mathcal{A}_n)^o + \mathfrak{J}_\omega$ is invertible in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_\omega$ for all $\omega \in [-1,1]$.*

Lemma 5.3. *The cosets $(\mathcal{M}_n^\tau \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n)^o$, $1 \le k \le m_-$ are contained in $\mathfrak{J}_\omega$ for $-1 < \omega \le 1$ and the cosets $(\mathcal{M}_n^\tau \mathcal{H}_{n,k}^{+,\tau} \mathcal{L}_n)^o$, $1 \le k \le m_+$ are contained in $\mathfrak{J}_\omega$ for $-1 \le \omega < 1$.*

Proof. We consider the case $\mathcal{H}_{n,k}^{-,\tau}$. The case $\mathcal{H}_{n,k}^{+,\tau}$ has to be handled in the same way. Let $-1 < \omega \le 1$ and let χ be a smooth function, which vanishes in some neighborhood of -1 and that satisfies $\chi(\omega) = 1$. Since $\chi \mathcal{B}_k^- : \mathbf{L}_\nu^2 \to \mathbf{C}[-1,1]$ is compact, the operator norm $\left\|(\mathcal{L}_n - \mathcal{M}_n^\tau)\chi \mathcal{B}_k^- \mathcal{L}_n\right\|_{\mathcal{L}(\mathbf{L}_\nu^2)}$ tends to zero for n tending to infinity. Due to the definition of the ideal $\mathfrak{J}$, we get $(\mathcal{M}_n^\tau \chi \mathcal{B}_k^- \mathcal{L}_n) \in \mathfrak{J}$. By Lemma 4.13 it follows that

$$\begin{aligned}\left(\mathcal{M}_n^\tau \chi \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n\right) &= \left(\mathcal{M}_n^\tau (1+\cdot)^{-\varepsilon} \chi \mathcal{L}_n\right) \cdot \\ &\quad \cdot \left(\mathcal{M}_n^\tau (1+\cdot)^{\varepsilon} \mathcal{L}_n \left[\mathcal{M}_n^\tau \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n - \mathcal{M}_n^\tau \mathcal{B}_k^- \mathcal{L}_n\right]\right) + \left(\mathcal{M}_n^\tau \chi \mathcal{B}_k^- \mathcal{L}_n\right)\end{aligned}$$

belongs to the ideal $\mathfrak{J}$. Thus,

$$\begin{aligned}\left(\mathcal{M}_n^\tau \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n\right)^o &= \left(\mathcal{M}_n^\tau \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n - \mathcal{M}_n^\tau \chi \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n\right)^o \\ &= \left(\mathcal{M}_n^\tau (1-\chi) \mathcal{L}_n \mathcal{M}_n^\tau \mathcal{H}_{n,k}^{-,\tau} \mathcal{L}_n\right)^o \in \mathfrak{J}_\omega\,.\end{aligned}$$

The lemma is proved. □

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As a consequence of Lemma 5.3, in case $-1<\omega<1$ the invertibility of the coset

$$\left(\mathcal{M}_n^\tau\Big(a\mathcal{I}+b\mathcal{S}+\sum_{k=1}^{m_-}\beta_k^-\mathcal{H}_{n,k}^{-,\tau}+\sum_{k=1}^{m_+}\beta_k^+\mathcal{H}_{n,k}^{+,\tau}\Big)\mathcal{L}_n\right)^o+\mathfrak{J}_\omega$$

is equivalent to the invertibility of $(\mathcal{M}_n^\tau(a\mathcal{I}+b\mathcal{S})\mathcal{L}_n)^o+\mathfrak{J}_\omega$. In the same manner as in [6, Corollary 5.13], we can state the following.

Lemma 5.4. *Let $(\mathcal{A}_n)\in\mathfrak{A}_0$. If the limit operator $\mathcal{W}^1(\mathcal{A}_n):\mathbf{L}^2_\nu\longrightarrow\mathbf{L}^2_\nu$ is Fredholm, then for all $\omega\in(-1,1)$, the coset $(\mathcal{A}_n)^o+\mathfrak{J}_\omega$ is invertible in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_\omega$.*

Now, let us investigate the invertibility of $(\mathcal{A}_n^\tau)^o+\mathfrak{J}_{\pm1}$ in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm1}$. To this end, we show that the invertibility of the limit operators $\mathcal{W}^3(\mathcal{A}_n^\tau)$ and $\mathcal{W}^4(\mathcal{A}_n^\tau)$ implies the invertibility of $(\mathcal{A}_n^\tau)^o+\mathfrak{J}_{+1}$ and $(\mathcal{A}_n^\tau)^o+\mathfrak{J}_{-1}$, respectively.

Lemma 5.5. *Let $\mathbf{R},\widetilde{\mathbf{R}}\in\operatorname{alg}\mathcal{T}(\mathbf{PC})$ and let*

$$\mathcal{A}_n^\tau=\mathcal{M}_n^\tau\left(a\mathcal{I}+b\mathcal{S}+\sum_{k_0=1}^{m_-}\beta_{k_0}^-\mathcal{H}_{n,k_0}^{-,\tau}+\sum_{k_0=1}^{m_+}\beta_{k_0}^+\mathcal{H}_{n,k_0}^{+,\tau}\right)\mathcal{L}_n,$$

$\mathbf{S}:=\mathcal{W}^3(\mathcal{A}_n^\tau)$, and $\mathbf{T}:=\mathcal{W}^4(\mathcal{A}_n^\tau)$. Then $\mathbf{S},\mathbf{T}\in\operatorname{alg}\mathcal{T}(\mathbf{PC})$, and the following hold:

(a) *We have $\left(\mathbf{R}_n^3\widetilde{\mathbf{R}}_n^3-[\mathbf{R}\widetilde{\mathbf{R}}]_n^3\right)^o\in\mathfrak{J}_{+1}$ and $\left(\mathbf{R}_n^4\widetilde{\mathbf{R}}_n^4-[\mathbf{R}\widetilde{\mathbf{R}}]_n^4\right)^o\in\mathfrak{J}_{-1}$.*
(b) *If $\mathbf{R}$ is invertible, then the coset $\left([\mathbf{R}^{-1}]_n^{3/4}\right)^o+\mathfrak{J}_{\pm1}$ is the inverse of $\left(\mathbf{R}_n^{3/4}\right)^o+\mathfrak{J}_{\pm1}$ in $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm1}$.*
(c) *We have $\left(\mathbf{S}_n^3\right)^o+\mathfrak{J}_1=(\mathcal{A}_n^\tau)^o+\mathfrak{J}_1$ and $\left(\mathbf{T}_n^4\right)^o+\mathfrak{J}_{-1}=(\mathcal{A}_n^\tau)^o+\mathfrak{J}_{-1}$.*

Proof. That $\mathbf{S}$ and $\mathbf{T}$ belong to $\operatorname{alg}\mathcal{T}(\mathbf{PC})$ is a consequence of Propositions 4.18, 4.19 and of [1, Lemma 3.12]. The proof of (a) is the same as the proof of the analogous assertion in [6, Lemma 5.14], which is only based on the fact, that $\mathfrak{J}_{\pm1}$ is generated by $\mathfrak{T}_{\pm1}$ (cf. the definition of $\mathfrak{J}_\omega$ before Proposition 5.2). Assertion (b) is an immediate consequence of (a). Let us prove (c) for the operator $\mathbf{T}$ in case of $\tau=\sigma$. The proof of the other cases is analogous. If $\mathcal{A}_n^\sigma=\mathcal{M}_n^\sigma(a\mathcal{I}+b\mathcal{S})\mathcal{L}_n$, assertion (c) was proved in [6, Lemma 5.16]. Hence, due to Lemma 4.12, it remains to show that

$$\left([\mathbf{A}_{k_0}]_n^4-\mathcal{M}_n^\sigma\mathcal{H}_{n,k_0}^{-,\sigma}\mathcal{L}_n\right)^o\in\mathfrak{J}_{-1}$$

is true. Define $\mathbf{B}_n$ as the matrix

$$\left[\frac{2}{\pi\mathbf{i}}\frac{\left(j+\frac12\right)^{2k_0-1}}{\left[\left(k+\frac12\right)^2+\left(j+\frac12\right)^2\right]^{k_0}}-\sqrt{1-x_{n-k,n}^\sigma}\,s_{n-j,n-k}^{(n)}\left[b_{n-j,n-k}^{(n)}\right]^{k_0-1}\right]_{j,k=0}^{n-1},$$

where

$$s_{jk}^{(n)} = \frac{1}{n\mathbf{i}} \frac{\sqrt{1+x_{jn}^\tau}}{x_{kn}^\tau + x_{jn}^\tau + 2} \quad \text{and} \quad b_{jk}^{(n)} = \frac{1+x_{jn}^\tau}{x_{kn}^\tau + x_{jn}^\tau + 2}\,.$$

Then, by definition of $[\mathbf{A}_{k_0}]_n^4$ and of $\widetilde{\mathcal{V}}_n^\sigma$ (cf. also (4.1)),

$$\mathbf{B}_n = \widetilde{\mathcal{V}}_n^\sigma \left([\mathbf{A}_{k_0}]_n^4 - \mathcal{M}_n^\sigma \mathcal{H}_{n,k_0}^{-,\sigma} \mathcal{L}_n \right) (\widetilde{\mathcal{V}}_n^\sigma)^{-1} \mathcal{P}_n\,.$$

In [1, Section 8], it is proved that

$$\begin{aligned} &\left\| \left([\mathbf{A}_{k_0}]_n^4 - \mathcal{M}_n^\sigma \mathcal{H}_{n,k_0}^{-,\sigma} \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{-1}} \\ &\quad = \left\| \left((\widetilde{\mathcal{V}}_n^\sigma)^{-1} \mathbf{B}_n \widetilde{\mathcal{V}}_n^\sigma \mathcal{L}_n \right)^o + \mathfrak{J}_{-1} \right\|_{(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{-1}} = 0\,. \end{aligned}$$

Thus, $\left([\mathbf{A}_{k_0}]_n^4 \right)^o + \mathfrak{J}_{-1} = \left(\mathcal{M}_n^\sigma \mathcal{H}_{n,k_0}^{-,\sigma} \mathcal{L}_n \right)^o + \mathfrak{J}_{-1}$. $\square$

Since we know that, for the generating sequences of $\mathfrak{A}_0$, the limit operators with $t \in \{3,4\}$ belong to $\operatorname{alg}\mathcal{T}(\mathbf{PC})$ (cf. Lemma 5.5) and since the mappings $\mathcal{W}^{3/4} : \mathfrak{F} \longrightarrow \mathfrak{L}(\ell^2)$ are continuous *-homomorphisms (see [6, Corollary 2.4]), we have $\mathcal{W}^{3/4}(\mathcal{A}_n) \in \operatorname{alg}\mathcal{T}(\mathbf{PC})$ if $(\mathcal{A}_n) \in \mathfrak{A}_0$. Thus, by Lemma 5.5 and the closedness of $\mathfrak{J}_{\pm 1}$, we get the following corollary.

Corollary 5.6. *Let $(\mathcal{A}_n) \in \mathfrak{A}_0$. Then, the invertibility of $\mathcal{W}^3(\mathcal{A}_n)$ and $\mathcal{W}^4(\mathcal{A}_n)$ implies, respectively, the invertibility of $(\mathcal{A}_n)^o + \mathfrak{J}_{+1}$ and $(\mathcal{A}_n)^o + \mathfrak{J}_{-1}$ in the algebra $(\mathfrak{A}/\mathfrak{J})/\mathfrak{J}_{\pm 1}$.*

Now, we are able to formulate the stability theorem for sequences of the algebra $\mathfrak{A}_0$, in particular for the quadrature method $(\mathcal{A}_n^\tau)$ given by (4.13). Indeed, with the help of Proposition 4.19, Lemma 5.4, Corollary 5.6, and the local principle of Allan and Douglas we can state the following theorem.

Theorem 5.7. *A sequence $(\mathcal{A}_n) \in \mathfrak{A}_0$ is stable if and only if all operators $\mathcal{W}^t(\mathcal{A}_n) : \mathbf{X}^{(t)} \longrightarrow \mathbf{X}^{(t)}$, $t = 1,2,3,4$ are invertible.*

We set

$$\mathbf{A}_- := a(-1)\mathbf{I} - b(-1)\tilde{\mathbf{S}} + \mathbf{A} \quad \text{and} \quad \mathbf{A}_+^\tau := a(1)\mathbf{I} + b(1)\mathbf{S}^\tau + \mathbf{A}^\tau.$$

Moreover, we define the curves

$$\begin{aligned} \Gamma_- &= \left\{ a(-1) + b(-1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4} + \mathbf{i}\xi]\right) : -\infty < \xi < \infty \right\} \\ &\quad \cup \left\{ a(-1) + b(-1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4} - \mathbf{i}\xi]\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^-(\tfrac{3}{4} + \mathbf{i}\xi) : -\infty \le \xi \le \infty \right\}, \end{aligned}$$

$$\Gamma_+^\sigma = \left\{a(1) + b(1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4} - \mathbf{i}\xi]\right) : -\infty < \xi < \infty\right\}$$
$$\cup \left\{a(1) + b(1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4} + \mathbf{i}\xi]\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+(\tfrac{1}{4} + \mathbf{i}\xi) : -\infty \le \xi \le \infty\right\},$$

$$\Gamma_+^\mu = \left\{a(1) - b(1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4} + \mathbf{i}\xi]\right) : -\infty < \xi < \infty\right\}$$
$$\cup \left\{a(1) + b(1)\mathbf{i}\cot\left(\pi[\tfrac{1}{4} + \mathbf{i}\xi]\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+(\tfrac{1}{4} + \mathbf{i}\xi) : -\infty \le \xi \le \infty\right\},$$

where

$$\widehat{\mathbf{h}}_k^\pm(\beta - \mathbf{i}t) = \binom{\beta - \mathbf{i}t + k - 2}{k-1} \frac{(\mp 1)^k}{\sinh(\pi(\mathbf{i}\beta + t))}, \quad 1 - k < \beta < 1, \ t \in \mathbb{R}.$$

Let us recall the following proposition concerning the collocation method.

Proposition 5.8. *(See [1], Theorem 4.11.) Let $a, b \in \mathbf{PC}$ and*

$$\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2.$$

Then, the collocation method $(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n)$, $\tau \in \{\sigma, \mu\}$ is stable if and only if

(a) *the operator $\mathcal{A} \in \mathcal{L}(\mathbf{L}_\nu^2)$ is invertible,*
(b) *the closed curves Γ_- and Γ_+^τ do not contain the zero point and their winding numbers vanish,*
(c) *the null spaces of the operators $\mathbf{A}_- + \mathbf{K}$, $\mathbf{A}_+^\tau + \mathbf{K}^\tau \in \mathcal{L}(\ell^2)$ are trivial,*
(d) *in case $\tau = \mu$, the operator $a\mathcal{J}_2 - \mathbf{i}b\mathcal{J}_3\mathcal{V} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ (cf. the formula for $\mathcal{W}^2(\mathcal{A}_n^\tau)$ in Proposition 4.18) is invertible,*

where $\mathbf{K}, \mathbf{K}^\tau : \ell^2 \longrightarrow \ell^2$ are the compact operators from Proposition 4.18.

Remark 5.9. In [1, Theorem 4.11] the invertibility condition in (d) is replaced by $|a(1)| > |b(1)|$. Under the remaining conditions of Proposition 5.8, this condition is sufficient for the invertibility of the operator $a\mathcal{I} - \mathbf{i}b\mathcal{J}_3\mathcal{V} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$. But it turns out, that it is not necessary, if $\beta_k^+ \neq 0$ for some index k.

Since, due to Lemma 4.9, Lemma 4.11, Lemma 4.12, and Lemma 4.17, we have that $\mathcal{W}^t(\mathcal{A}_n^\tau) = \mathcal{W}^t(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n)$, $t \in \{1, 2\}$, and that $\mathcal{W}^t(\mathcal{A}_n^\tau) - \mathcal{W}^t(\mathcal{M}_n^\tau \mathcal{A} \mathcal{L}_n)$, $t \in \{3, 4\}$ are compact operators, we deliver our final theorem with the help of Theorem 5.7 and Proposition 5.8.

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Theorem 5.10. *The quadrature method $(\mathcal{A}_n^\tau)$, $\tau \in \{\sigma, \mu\}$ is stable if and only if*

(a) *the operator $\mathcal{A} \in \mathcal{L}(\mathbf{L}_\nu^2)$ is invertible,*
(b) *the closed curves Γ_- and Γ_+^τ do not contain the zero point and their winding numbers vanish,*
(c) *the null spaces of the operators $\mathbf{A}_-, \mathbf{A}_+^\tau \in \mathcal{L}(\ell^2)$ are trivial,*
(d) *in case $\tau = \mu$, the operator $a\mathcal{J}_2 - \mathbf{i}b\mathcal{J}_3\mathcal{V} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ is invertible.*

Condition (b) in Theorem 5.10 is equivalent to the Fredholmness with index 0 of the third and fourth limit operators. Together with condition (c), this implies the invertibility of these operators. In case of $\beta_k^\pm = 0$ for all k (i.e., the Mellin operators do not occur), then (b) implies (c) (cf. [6, Corollary 4.9]).

For the investigation of the invertibility of the operator $\mathcal{A}$ we refer to the following Proposition formulated in [1, Theorem 3.5] and basing on respective results from [4,9,10].

Proposition 5.11. *Let $a, b \in \mathbf{PC}$ and $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^- + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{B}_k^+ : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$.*

(a) *The operator $\mathcal{A}$ is Fredholm if and only if*
  - *For any $x \in (-1, 1)$, there hold $a(x \pm 0) + b(x \pm 0) \neq 0$ and $a(x \pm 0) - b(x \pm 0) \neq 0$ as well as $a(\pm 1) + b(\pm 1) \neq 0$ and $a(\pm 1) - b(\pm 1) \neq 0$.*
  - *If a or b has a jump at $x \in (-1, 1)$, then there holds*

$$\lambda \frac{a(x+0) + b(x+0)}{a(x+0) - b(x+0)} + (1 - \lambda) \frac{a(x-0) + b(x-0)}{a(x-0) - b(x-0)} \neq 0, \quad 0 \le \lambda \le 1.$$

  - *For $x = \pm 1$, there hold*

$$a(1) + b(1)\mathbf{i}ct\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+ \left(\tfrac{1}{4} - \mathbf{i}\xi\right) \neq 0, \quad -\infty < \xi < \infty,$$

  and

$$a(-1) + b(-1)\mathbf{i}ct\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^- \left(\tfrac{3}{4} - \mathbf{i}\xi\right) \neq 0, \quad -\infty < \xi < \infty.$$

(b) *If $\mathcal{A}$ is Fredholm and if the coefficients a and b have finitely many jumps, then the Fredholm index of $\mathcal{A} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ is equal to minus the winding number of the closed curve*

$$\Gamma_{\mathcal{A}} := \Gamma_- \cup \Gamma_1 \cup \Gamma_1' \cup \cdots \cup \Gamma_N \cup \Gamma_N' \cup \Gamma_{N+1} \cup \Gamma_+$$

with the orientation given by the subsequent parametrization. Here, N stands for the number of discontinuity points x_i, $i = 1, \ldots, N$, of the functions a and b chosen *such that $x_0 := -1 < x_1 < \cdots < x_N < x_{N+1} := 1$. Using these x_i, the curves Γ_i, $i = 1, \dots, N+1$, and Γ_i', $i = 1, \dots, N$ are given by*

$$\Gamma_i := \left\{ \frac{a(y) + b(y)}{a(y) - b(y)} \ : \ x_{i-1} < y < x_i \right\},$$

$$\Gamma_i' := \left\{ \lambda \frac{a(x_i + 0) + b(x_i + 0)}{a(x_i + 0) - b(x_i + 0)} + (1 - \lambda) \frac{a(x_i - 0) + b(x_i - 0)}{a(x_i - 0) - b(x_i - 0)} \ : \ 0 \le \lambda \le 1 \right\}.$$

The curves Γ_+ and Γ_-, connecting the point 1 with one of the end points of Γ_1 and Γ_{N+1}, are given by

$$\left\{ \frac{a(1) + b(1)\mathbf{i}\cot\left(\frac{\pi}{4} - \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbf{h}}_k^+ (\frac{1}{4} - \mathbf{i}\xi)}{a(1) - b(1)} \ : \ -\infty \le \xi \le \infty \right\}$$

and

$$\left\{ \frac{a(-1) + b(-1)\mathbf{i}\cot\left(\frac{\pi}{4} + \mathbf{i}\pi\xi\right) + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbf{h}}_k^- (\frac{3}{4} - \mathbf{i}\xi)}{a(-1) - b(-1)} \ : \ -\infty \le \xi \le \infty \right\},$$

respectively.

(c) *If $\mathcal{A}$ is Fredholm and if $m_- = 0$ or $m_+ = 0$, then $\mathcal{A}$ is one-sided invertible.*

6. Matrix structure and fast summation

To compute the solution $u_n \in \operatorname{im} \mathcal{L}_n$ of equation (2.7) we can solve the corresponding system of linear equations

$$\widetilde{\mathbb{A}}_n \widetilde{\xi}_n = \widetilde{\eta}_n \,, \tag{6.1}$$

where $\widetilde{\xi}_n = \left[\, \xi_{kn} \, \right]_{k=1}^n$ is the vector of the coefficients of u_n in the basis $\left\{ \widetilde{\ell}_{kn}^\tau : k = 1, \dots, n \right\}$ of the space $\operatorname{im} \mathcal{L}_n$, where $\widetilde{\eta}_n = \left[\, f(x_{jn}^\tau) \, \right]_{j=1}^n$, and where

$$\widetilde{\mathbb{A}}_n = \left[\left(\mathcal{A}_n \widetilde{\ell}_{kn}^\tau \right) (x_{jn}^\tau) \right]_{j,k=1}^n \ \text{with} \ \mathcal{A}_n = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{H}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{H}_{n,k}^{+,\tau} \,.$$

Since the system $\left\{ (\omega_n^\tau)^{-1} (1 + x_{kn}^\tau)^{-\frac{1}{2}} \widetilde{\ell}_{kn}^\tau : k = 1, \dots, n \right\}$ forms an orthonormal basis in $\operatorname{im} \mathcal{L}_n$, we define $\mathbb{A}_n : \operatorname{im} \mathcal{P}_n \longrightarrow \operatorname{im} \mathcal{P}_n$ as

$$\mathbb{A}_n = \left[\sqrt{\tfrac{1 + x_{jn}^\tau}{1 + x_{kn}^\tau}} \left(\mathcal{A}_n \widetilde{\ell}_{kn}^\tau \right) (x_{jn}^\tau) \right]_{j,k=1}^n = \mathcal{V}_n^\tau \mathcal{M}_n^\tau \mathcal{A}_n \mathcal{L}_n (\mathcal{V}_n^\tau)^{-1}$$

and, instead of (6.1), solve the preconditioned (in case of stability of the collocation-quadrature method) system

$$\mathbb{A}_n \xi_n = \eta_n \,, \tag{6.2}$$

where $\mathbb{A}_n = \mathbb{D}_n^{-,\tau} \widetilde{\mathbb{A}}_n \left(\mathbb{D}_n^{-,\tau}\right)^{-1}$, $\eta_n = \mathbb{D}_n^{-,\tau} \widetilde{\eta}_n$, and $\xi_n = \mathbb{D}_n^{-,\tau} \widetilde{\xi}_n$ with $\mathbb{D}_n^{\pm,\tau} = \operatorname{diag}\left[\sqrt{1 \mp x_{jn}^\tau}\right]_{j=1}^n$. For solving (6.2) by an iteration method (for example, the Krylov subspace method CGNR), we are interested in a fast matrix–vector multiplication by the matrix $\mathbb{A}_n$. For this, we study the structure of this matrix in more detail and set $\mathbb{D}_n^{a,\tau} = \operatorname{diag}\left[a(x_{jn}^\tau)\right]_{j=1}^n$,

$$\mathbb{S}_n^\tau = \left[\sqrt{\tfrac{1+x_{jn}^\tau}{1+x_{kn}^\tau}} \left(\mathcal{S}\widetilde{\ell}_{kn}^\tau\right)(x_{jn}^\tau)\right]_{j,k=1}^n \,,$$

and

$$\mathbb{H}_{n,k}^{\pm,\tau} = \left[\sqrt{\tfrac{1+x_{jn}^\tau}{1+x_{kn}^\tau}} \left(\mathcal{H}_{n,k}^{\pm,\tau}\widetilde{\ell}_{kn}^\tau\right)(x_{jn}^\tau)\right]_{j,k=1}^n \,,$$

such that

$$\mathbb{A}_n = \mathbb{D}_n^{a,\tau} + \mathbb{D}_n^{b,\tau}\mathbb{S}_n^\tau + \sum_{k=1}^{m_-} \beta_k^- \mathbb{H}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \mathbb{H}_{n,k}^{+,\tau} \,. \tag{6.3}$$

From [1, Section 6] we infer that

$$\mathbb{S}_n^\sigma = \frac{2\mathbf{i}}{n}\, \mathbb{C}_n^4 \mathbb{S}_n^4 \quad \text{and} \quad \mathbb{S}_n^\mu = \frac{4\mathbf{i}}{2n+1}\, \mathbb{C}_n^6 \mathbb{S}_n^7 \,, \tag{6.4}$$

where

$$\mathbb{S}_n^4 = \left[\sin\frac{(2j-1)(2k-1)\pi}{4n}\right]_{j,k=1}^n \,, \quad \mathbb{S}_n^7 = \left[\sin\frac{(2j-1)k\pi}{2n+1}\right]_{j,k=1}^n \,,$$

and

$$\mathbb{C}_n^4 = \left[cs\frac{(2j-1)(2k-1)\pi}{4n}\right]_{j,k=1}^n \,, \quad \mathbb{C}_n^6 = \left[cs\frac{j(2k-1)\pi}{2n+1}\right]_{j,k=1}^n$$

are discrete sine and cosine transforms, which can be applied to a vector of length n by $\mathcal{O}(n \log n)$ complexity. By definition of $\mathcal{H}_{n,k_0}^{\pm,\tau}$ (see (2.6)) we get

$$\mathbb{H}_{n,k_0}^{\pm,\tau} = \gamma_n^\tau \mathbb{D}_n^{-,\tau} \left(\mathbb{D}_n^{\pm,\tau}\right)^{2(k_0-1)} \mathbb{B}_{n,k_0}^{\pm,\tau} \mathbb{D}_n^{+,\tau} \tag{6.5}$$

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with $\gamma_n^\sigma = \frac{1}{\mathbf{i}n}$, $\gamma_n^\mu = \frac{1}{\mathbf{i}(n+\frac{1}{2})}$, and

$$\mathbb{B}_{n,k_0}^{\pm,\tau} = \left[\frac{1}{\left(x_{kn}^\tau + x_{jn}^\tau \mp 2\right)^{k_0}} \right]_{j,k=1}^n .$$

If we define $x_k^\pm = 7(\pm 1 - x_{kn}^\tau)/64$ and $y_j^\pm = 7(x_{jn}^\tau \mp 1)/64$, then we can write

$$\mathbb{B}_{n,k_0}^{\pm,\tau} = \left(\frac{7}{64}\right)^{k_0} \left[K_{k_0}(y_j^\pm - x_k^\pm) \right]_{j,k=1}^n \quad \text{with} \quad K_{k_0}(x) = \frac{1}{x^{k_0}} .$$

For the matrix–vector multiplication by $\mathbb{H}_{n,k_0}^{\pm,\tau}$, this enables us to apply the method of fast summation at nonequispaced nodes by NFFTs. In [11] a fast summation method for the approximative multiplication of a vector by matrices of the form $\mathbb{B}_{n,k_0}^{\pm,\tau}$ is presented. The method is based on splitting the kernel $K_{k_0}(x)$ in a nearfield $K_{\mathrm{NE}}(x)$ with $|x| = |y_j^\pm - x_k^\pm| \le \frac{\alpha}{n}$ and a smooth one-periodic function $K_{\mathrm{RF}}(x)$ (see e.g. [12, Fig. 3.1]) which can be approximated by a Fourier series and realized in $\mathcal{O}(n \log n)$, see [11, Alg. 3.1]. In our special case we know that the distance between the nodes $x_k^\pm$ and $y_j^\pm$ is proportional $1/n$ in the middle of the interval and proportional to $1/n^2$ near the borders. However, the last estimate is too pessimistic because the node distance of $1/n^2$ only applies quite close to the border $x = 0$. Using estimates like in [12, Theorem 4.1], we see that the nearfield evaluation is also bounded by $\mathcal{O}(n \log n)$.

If the method is stable, we know that, for sufficiently large n, the condition number $\mathrm{cond}(\mathbb{A}_n)$ is bounded. We apply the standard estimate for the convergence of the conjugate gradient method (see [13, p. 289], in order to see that a bounded number of iterations is sufficient to approximate the solution within a certain prescribed accuracy. Therefore, we can solve the system (6.2) with $\mathcal{O}(n \log n)$ arithmetical operations.

7. Application to the notched half plane problem

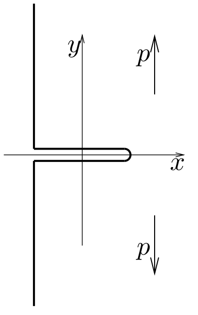
In this section we apply the collocation–quadrature method to the integral equation of the notched half plane problem (cf. [14, Section 37a], [4, Section 14], [15, Equ. (1.8)])

$$\frac{1}{\pi} \int_{-1}^{1} \left[\frac{1}{y-x} - \frac{1}{2+y+x} + \frac{6(1+x)}{(2+y+x)^2} - \frac{4(1+x)^2}{(2+y+x)^3} \right] u(y)\, dy = f(x) , \qquad (7.1)$$

$-1 < x < 1$, and compare the results with the results obtained in [15, Section 5] using the collocation method (cf. also [1, Section 6]). The right hand side $f(x) = -\frac{1+\kappa_0}{2\mu_0} p$ in (7.1) describes the tensile forces subjecting the elastic half plane at infinity perpendicular to the line of the straight crack of normalized length 2, which is assumed to be free of external forces and to end at the boundary of the half plane at right angle (see the figure). The elastic constants κ_0 and μ_0 are the shearing modulus and the Muskhelishvilli constant, respectively. The unknown function $u(x)$ is the derivative of the crack opening displacement. For practical purposes, the stress intensity factor at point $x = 1$ defined by

$$k_1(1) := -\frac{2\mu_0}{1+\kappa_0} \lim_{x\to 1-0} \left[u(x)\sqrt{2(1-x)}\right] \tag{7.2}$$

is of main interest in fracture mechanics (cf. [16, (75)]).



It is known [4, Theorem 14.1] that the solution $u(x)$ of (7.1) has a singularity of the form $(1-x)^{-\frac{1}{2}}$. For that reason we use a collocation–quadrature method in which the approximate solution has the form $u_n(x) = \nu(x)p_n(x)$ with a polynomial $p_n(x)$ of degree less than n. Hence, we consider equation (7.1) or, more general, equation (2.1) in $\mathbf{L}^2_\mu$ and search for an approximate solution of the form

$$u_n(x) = \sum_{k=1}^{n} \xi_{kn}\widehat{\ell}^\tau_{kn} \quad \text{with} \quad \widehat{\ell}^\tau_{kn} = \frac{\nu(x)\ell^\tau_{kn}(x)}{\nu(x^\tau_{kn})}\,.$$

Defining the Fourier projections

$$\widehat{\mathcal{L}}_n : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu\,, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widehat{p}_j\rangle_\mu\, \widehat{p}_j\,,$$

where $\widehat{p}_j(x) = \nu(x)R_n(x)$, and the weighted interpolation operator $\widehat{\mathcal{M}}^\tau_n = \nu\mathcal{L}^\tau_n\nu^{-1}\mathcal{I}$ we determine $u_n \in \operatorname{im}\widehat{\mathcal{L}}_n$ by solving, instead of (2.7), the collocation–quadrature equations

$$\widehat{\mathcal{A}}^\tau_n u_n := \widehat{\mathcal{M}}^\tau_n \left(a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-}\beta^-_k\mathcal{H}^{-,\tau}_{n,k} + \sum_{k=1}^{m_+}\beta^+_k\mathcal{H}^{+,\tau}_{n,k}\right)\widehat{\mathcal{L}}_n u_n = \widehat{\mathcal{M}}^\tau_n f \tag{7.3}$$

for $\tau = \sigma$ and for $\tau = \nu$. Analogously to [1, Lemma 5.1] we can prove the following lemma, which enables us to apply Theorem 5.10 also in case of the method (7.3).

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Lemma 7.1. *The collocation–quadrature method* $\left(\widehat{\mathcal{A}}_n^\tau\right)$ *given by* (7.3) *is stable in* $\mathbf{L}^2_\mu$ *if and only if the method* $(\mathcal{A}_n^\rho)$ *with*

$$\mathcal{A}_n^\rho := \mathcal{M}_n^\rho \left(\widetilde{a}\mathcal{I} - \widetilde{b}\mathcal{S} + \sum_{k=1}^{m_-} (-1)^k \beta_k^- \mathcal{H}_{n,k}^{+,\rho} + \sum_{k=1}^{m_+} (-1)^k \beta_k^+ \mathcal{H}_{n,k}^{-,\rho} \right) \mathcal{L}_n \,,$$

where $\widetilde{a}(x) := a(-x)\,,\ \widetilde{b}(x) := b(-x)$*, is stable in* $\mathbf{L}^2_\nu$*, where* $\rho = \sigma$ *if* $\tau = \sigma$ *and* $\rho = \mu$ *if* $\tau = \nu$*.*

Remark. Of course, equivalently one can transform equation (7.1) (considered in $\mathbf{L}^2_\mu$) with the help of the isometric isomorphism $\mathcal{J} : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\nu$, $u(x) \mapsto u(-x)$ into an equation in the space $\mathbf{L}^2_\nu$ and then apply method (2.7) to the transformed equation. Since, in the literature, the integral equation of the notched half plane problem is often written in the form (7.1), we prefer to use the method (7.3) directly for equation (7.1), where $m_- = 3$, $m_+ = 0$, and

$$a = 0\,,\ b = \mathbf{i}\,,\ \beta_1^- = -\mathbf{i}\,,\ \beta_2^- = 6\mathbf{i}\,,\ \beta_3^- = -4\mathbf{i}\,.$$

For that reason, in what follows, we give the respective formulas concerned with the collocation–quadrature method (7.3) and analogous to the formulas given in Section 6.

We find the solution $u_n \in \operatorname{im} \widehat{\mathcal{L}}_n$ by solving the (preconditioned) linear system (cf. Section 6)

$$\widehat{\mathbb{A}}_n \xi_n = \eta_n := \left[\sqrt{1 - x_{jn}^\tau}\, f(x_{jn}^\tau) \right]_{j=1}^n \,, \tag{7.4}$$

where the matrix $\widehat{\mathbb{A}}_n = \left[\sqrt{\frac{1-x_{jn}^\tau}{1-x_{kn}^\tau}} \left(\mathcal{A}_n \widehat{\ell}_{kn}^\tau \right) (x_{jn}^\tau) \right]_{j,k=1}^n$ with $\mathcal{A}_n = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{H}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \mathcal{H}_{n,k}^{+,\tau}$ has the structure (cf. (6.3))

$$\widehat{\mathbb{A}}_n = \mathbb{D}_n^{a,\tau} + \mathbb{D}_n^{b,\tau} \widehat{\mathbb{S}}_n^\tau + \sum_{k=1}^{m_-} \beta_k^- \widehat{\mathbb{H}}_{n,k}^{-,\tau} + \sum_{k=1}^{m_+} \beta_k^+ \widehat{\mathbb{H}}_{n,k}^{+,\tau} \,.$$

Instead of (6.4) and (6.5), we have, using the same method as in [1, Section 6],

$$\widehat{\mathbb{S}}_n^\sigma = -\frac{2\mathbf{i}}{n} \mathbb{S}_n^4 \mathbb{C}_n^4 \,, \quad \widehat{\mathbb{S}}_n^\nu = -\frac{4\mathbf{i}}{2n+1} \mathbb{S}_n^8 \mathbb{C}_n^8 \,,$$

and

$$\widehat{\mathbb{H}}_{n,k_0}^{\pm,\tau} = \gamma_n^\tau \mathbb{D}_n^{+,\tau} \left(\mathbb{D}_n^{\pm,\tau} \right)^{2(k_0-1)} \mathbb{B}_{n,k_0}^{\pm,\tau} \mathbb{D}_n^{-,\tau}$$

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with $\gamma_n^\nu = \gamma_n^\mu$ and the trigonometric transforms (for $\mathbb{S}_n^4$ and $\mathbb{C}_n^4$, see Section 6)

$$\mathbb{S}_n^8 = \left[\sin \frac{(2j-1)(2k-1)\pi}{2(2n+1)} \right]_{j,k=1}^n$$

as well as

$$\mathbb{C}_n^8 = \left[cs \frac{(2j-1)(2k-1)\pi}{2(2n+1)} \right]_{j,k=1}^n .$$

In practise, we realize these transforms with the help of the FFTW with a complexity of $\mathcal{O}(n \log n)$ (see http://fftw.org/fftw_doc).

Concerning the stability of the method (7.3) applied to equation (7.1) we have to verify the conditions of Theorem 5.10 for the operator

$$\mathcal{A} = \mathcal{S} - \mathcal{B}_1^+ - 6\mathcal{B}_2^+ - 4\mathcal{B}_3^+ \tag{7.5}$$

(cf. Lemma 7.1, see also [1, pp. 216, 217]). Denote, for $z_1, z_2 \in \mathbb{C}$, by $\gamma_{\ell/r}[z_1, z_2]$ the half circle line from z_1 to z_2 lying on the left/right of the segment $[z_1, z_2]$. In view of Proposition 5.11, for the invertibility of $\mathcal{A} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ we have to show that the closed curve

$$\begin{aligned}\Gamma_{\mathcal{A}} &= \left\{ -\mathbf{i} \cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \\ &\quad \cup \Big\{ -\mathbf{i} \cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) + \widehat{\mathbf{h}}_1^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) \\ &\quad + 6\widehat{\mathbf{h}}_2^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) + 4\widehat{\mathbf{h}}_3^+\left(\tfrac{1}{4} - \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \Big\} \\ &= \gamma_\ell[1, -1] \cup \Gamma_0\end{aligned}$$

does not contain 0 and that its winding number is equal to zero, what can be seen from Fig. 1. Note that Γ_0 is a distorted arc $\gamma_r[-1, 1]$. Obviously, condition (d) of Theorem 5.10 is not fulfilled, since in the present case we have $a = 0$ and since $\mathcal{V} : \mathbf{L}_\nu^2 \longrightarrow \mathbf{L}_\nu^2$ is not invertible. This implies that the sequence $\left(\widehat{\mathcal{A}}_n^\nu\right)$ for the collocation w.r.t. the Chebyshev nodes of third kind is not stable. That's why we concentrate on the case $\tau = \sigma$ (cf. also Tables 4–6). Due to condition (b) of Theorem 5.10 we have to consider the curves

$$\Gamma_- = \left\{ \mathbf{i} \cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \cup \left\{ \mathbf{i} \cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\}$$

and

$$\begin{aligned}\Gamma_+^\sigma &= \left\{ \cot\left(\tfrac{\pi}{4} - \mathbf{i}\pi\xi\right) : -\infty \le \xi \le \infty \right\} \\ &\quad \cup \Big\{ \cot\left(\tfrac{\pi}{4} + \mathbf{i}\pi\xi\right) - \widehat{\mathbf{h}}_1^+\left(\tfrac{1}{4} + \mathbf{i}\xi\right) \\ &\quad - 6\widehat{\mathbf{h}}_2^+\left(\tfrac{1}{4} + \mathbf{i}\xi\right) - 4\widehat{\mathbf{h}}_3^+\left(\tfrac{1}{4} + \mathbf{i}\xi\right) : -\infty \le \xi \le \infty \Big\}.\end{aligned}$$

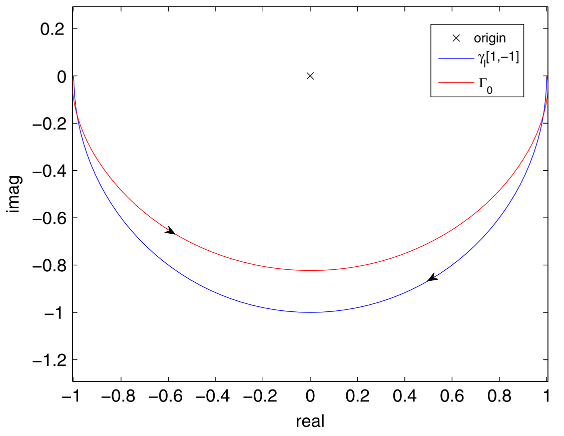

Fig. 1. $\Gamma_{\mathcal{A}}$ for the operator (7.1).

Since Γ_- is the union of two half circles (from -1 to $+1$ and back) lying on the same side of the diameter, the winding number of this curve is equal to zero and, since $\Gamma_+^\sigma = -\Gamma_{\mathcal{A}}$ (with reverse direction), the curve Γ_+^σ has also winding number zero. Hence, the operators $\mathbf{A}_-, \mathbf{A}_+^\sigma \in \mathcal{L}(\ell^2)$ are Fredholm with index zero. Concerning condition (c) of Theorem 5.10, our conjecture is that it is satisfied in case $\tau = \sigma$, which is confirmed by the sequence of condition numbers presented in Table 1.

In the following Tables 1–6 we present numerical results (for $f(x) \equiv 1$ in (7.1)) obtained by applying the Krylov subspace method CGNR to the linear systems of the collocation method, the collocation–quadrature method, and the collocation–quadrature method combined with the fast summation method. In these tables, M denotes the number of iterations needed to obtain a residual norm which is smaller than 10^{-12} times the initial residual norm. Moreover, by $k_1(1)$ and $d(-1)$ we denote (the numerical values of) the normalized stress intensity factor at the point $+1$ and the normalized crack opening displacement at the point -1, respectively,

$$k_1(1) = \lim_{x \to 1-0} u(x)\sqrt{1-x} \quad \text{and} \quad d(-1) = -\frac{1}{2}\int_{-1}^{1} u(x)\, dx$$

(cf. (7.2) and recall that we have chosen $f \equiv 1$). Thereby, $k_1(1)$ is calculated by interpolating the computed approximate values of $u(x)\sqrt{1-x}$ at x_{1n}^τ, x_{2n}^τ, and x_{3n}^τ, while, for $d(-1)$, we use

$$d(-1) \approx -\frac{1}{2}\sum_{k=1}^{n} \frac{\lambda_{kn}^\tau u(x_{kn}^\tau)}{\tau(x_{kn}^\tau)}\,.$$

The condition numbers in Tables 4–6 confirm our theoretical results that the methods in case $\tau = \nu$ are not stable. Otherwise, the results shown in Tables 1–3 (Table 1 was

Table 1
Collocation for (7.1) with $\tau = \sigma$.

n	M	cond$(\mathbb{A}_n)$	$k_1(1)$	$d(-1)$
16	9	1.3501	1.12185858	1.45420437
32	9	1.3727	1.12158581	1.45435329
64	10	1.3881	1.12153563	1.45433123
128	10	1.3990	1.12152529	1.45431106
256	10	1.4069	1.12152297	1.45430235
512	11	1.4129	1.12152243	1.45429926
1024	11	1.4174	1.12152230	1.45429826
2048	12	1.4204	1.12152227	1.45429795
4096	12	1.4233	1.12152226	1.45429786
8192	12	1.4255	1.12152226	1.45429783
16 384	13		1.12152226	1.45429782

Table 2
Collocation–quadrature (7.3) for (7.1) with $\tau = \sigma$ (red color – differences w.r.t. Table 1). (For interpretation of the references to color in this table legend, the reader is referred to the web version of this article.)

n	M	cond$(\mathbb{A}_n)$	$k_1(1)$	$d(-1)$
16	9	1.6681	1.12116268	1.44889890
32	9	1.6892	1.12141852	1.45273539
64	10	1.7031	1.12149464	1.45385436
128	10	1.7127	1.12151515	1.45417379
256	11	1.7196	1.12152045	1.45426352
512	11	1.7248	1.12152180	1.45428842
1024	11	1.7287	1.12152214	1.45429526
2048	12	1.7318	1.12152223	1.45429713
4096	12	1.7343	1.12152225	1.45429763
8192	13	1.7363	1.12152225	1.45429777
16 384	13	1.7379	1.12152225	1.45429781

Table 3
Collocation–quadrature (7.3) and fast summation for (7.1) with $\tau = \sigma$ (blue color – differences w.r.t. Table 2). (For interpretation of the references to color in this table legend, the reader is referred to the web version of this article.)

n	M	cond$(\mathbb{A}_n)$	$k_1(1)$	$d(-1)$
16	9	1.6681	1.12116300	1.44889894
32	9	1.6892	1.12141887	1.45273536
64	10	1.7031	1.12149499	1.45385435
128	10	1.7127	1.12151552	1.45417378
256	11	1.7196	1.12152084	1.45426351
512	11	1.7248	1.12152221	1.45428842
1024	11	1.7287	1.12152256	1.45429526
2048	12	1.7318	1.12152265	1.45429713
4096	12	1.7343	1.12152267	1.45429763
8192	13	1.7363	1.12152268	1.45429777
16 384	13	1.7379	1.12152269	1.45429781

already presented in [15, table on page 112]) suggest that the methods for $\tau = \sigma$ are stable, which is also confirmed by the fact that due to the above discussion, in this case, the conditions (a) and (b) of Proposition 5.8 and of Theorem 5.10 are fulfilled.

In Table 7 we present the CPU-time used for the Krylov subspace iteration by the different methods (CM – collocation method considered in [15] with complexity

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Table 4
Collocation for (7.1) with $\tau = \nu$ (blue color – differences w.r.t. Table 1). (For interpretation of the references to color in this table legend, the reader is referred to the web version of this article.)

n	M	$\mathrm{cond}(\mathbb{A}_n)$	$k_1(1)$	$d(-1)$
16	9	4.89	1.12056842	1.48973797
32	10	6.87	1.12127461	1.47403928
64	11	9.69	1.12145928	1.46479562
128	11	13.68	1.12150639	1.45973338
256	12	19.34	1.12151827	1.45706965
512	12	27.34	1.12152126	1.45569907
1024	13	38.65	1.12152201	1.45500273
2048	13	54.66	1.12152219	1.45465146

Table 5
Collocation–quadrature (7.3) for (7.1) with $\tau = \nu$ (red color – differences w.r.t. Table 4). (For interpretation of the references to color in this table legend, the reader is referred to the web version of this article.)

n	M	$\mathrm{cond}(\mathbb{A}_n)$	$k_1(1)$	$d(-1)$
16	9	5.02	1.12053009	1.48950896
32	10	7.07	1.12126441	1.47396049
64	11	9.97	1.12145667	1.46477090
128	11	14.09	1.12150572	1.45972598
256	12	19.91	1.12151811	1.45706750
512	12	28.15	1.12152122	1.45569846
1024	13	39.81	1.12152200	1.45500256
2048	13	56.30	1.12152219	1.45465144

Table 6
Collocation–quadrature (7.3) and fast summation for (7.1) with $\tau = \nu$ (blue color – differences w.r.t. Table 5). (For interpretation of the references to color in this table legend, the reader is referred to the web version of this article.)

n	M	$\mathrm{cond}(\mathbb{A}_n)$	$k_1(1)$	$d(-1)$
16	11	5.02	1.12053827	1.48946900
32	10	7.07	1.12135318	1.47391360
64	11	9.97	1.12157351	1.46473453
128	12	14.09	1.12163836	1.45970121
256	13	19.91	1.12166060	1.45705182
512	13	28.15	1.12167013	1.45568898
1024	14	39.81	1.12167534	1.45499701
2048	15	57.40	1.12167912	1.45464823

Table 7
CPU-time for the iteration process in seconds (case $\tau = \sigma$).

n	256	512	1024	2048	4096	8192	16 384
CM	0.07	0.32	1.49	6.51	30.46	162.46	856.36
CQM	0.03	0.10	0.38	1.60	6.34	27.57	109.74
CQFS	0.26	0.36	0.55	0.96	1.74	3.30	6.83

$\mathcal{O}(n^2)$, CQM – collocation–quadrature method (7.3) with complexity $\mathcal{O}(n^2)$, CQFS – collocation–quadrature method (7.3) with fast summation).

For CQFS, we used the software [17] for the kernels $1/x$, $1/x^2$ and $1/x^3$, where we choose the near field parameter $\frac{\alpha}{n} = \frac{8}{n}$, the size of the FFT as $2n$, and the Kaiser–Bessel window.

8. Error estimates

In this section we apply some results on the error of Lagrange interpolation in Sobolev like and in weighted $\mathbf{L}^p$ norms for proving error estimates for the collocation–quadrature method (7.3) in case $\tau = \sigma$ as well as for the respective collocation method

$$\widehat{\mathcal{M}}_n^\sigma \mathcal{A} \widehat{\mathcal{L}}_n u_n = \widehat{\mathcal{M}}_n^\sigma f \tag{8.1}$$

in case $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^-$. For this we introduce the Sobolev type subspaces $\mathbf{L}_\omega^{2,s}$ and $\widetilde{\mathbf{L}}_\mu^{2,s}$, $s \geq 0$ of $\mathbf{L}_\omega^2$ and $\mathbf{L}_\mu^2$ by (cf. [18,19])

$$\mathbf{L}_\omega^{2,s} = \left\{ f \in \mathbf{L}_\omega^2 : \sum_{n=0}^\infty (n+1)^{2s} \left| \langle f, p_n^\omega \rangle_\omega \right|^2 < \infty \right\}$$

and

$$\widetilde{\mathbf{L}}_\mu^{2,s} = \left\{ f \in \mathbf{L}_\mu^2 : \sum_{n=0}^\infty (n+1)^{2s} \left| \langle f, \nu R_n \rangle_\mu \right|^2 < \infty \right\},$$

equipped with the norms

$$\|f\|_{\omega,s} = \sqrt{\sum_{n=0}^\infty (n+1)^{2s} \left| \langle f, p_n^\omega \rangle_\omega \right|^2}$$

and

$$\|f\|_{\mu,s,\sim} = \sqrt{\sum_{n=0}^\infty (n+1)^{2s} \left| \langle f, \nu R_n \rangle_\mu \right|^2},$$

respectively. Here $\omega = v^{\alpha,\beta}$ with $\alpha, \beta > -1$ denotes a Jacobi weight and $(p_n^\omega)_{n=0}^\infty$ the respective system of orthonormal polynomials. It can be easily checked that, for $0 \leq s \leq t$ and a polynomial p_n of degree less than n,

$$\|p_n\|_{\omega,t} \leq n^{t-s} \|p_n\|_{\omega,s} \quad \text{and} \quad \|u_n\|_{\mu,t,\sim} \leq n^{t-s} \|u_n\|_{\mu,s,\sim} \ \text{ for } u_n \in \operatorname{im} \widehat{\mathcal{L}}_n \tag{8.2}$$

as well as

$$\left\| u - \widehat{\mathcal{L}}_n u \right\|_{\mu,s,\sim} \leq n^{s-t} \|u\|_{\mu,t,\sim}, \quad u \in \widetilde{\mathbf{L}}_\mu^{2,t}. \tag{8.3}$$

Moreover, due to the well known relation $\mathcal{S}\nu R_n = -\mathbf{i} P_n$, $n \in \mathbb{N}_0$,

$$\mathcal{S} \in \mathcal{L}(\widetilde{\mathbf{L}}_\mu^{2,s}, \mathbf{L}_\mu^{2,s}). \tag{8.4}$$

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Lemma 8.1. *(See [18], pp. 196, 197.) If $r \geq 0$ is an integer, then $u \in \mathbf{L}^{2,r}_{\omega}$ if and only if $u^{(k)}\varphi^k \in \mathbf{L}^2_{\omega}$ for $k = 0, 1, \ldots, r$. (Here $u^{(k)}$ has to be understood in the sense of a generalized derivative.) Moreover, the norms*

$$\|u\|_{\omega,r} \quad \text{and} \quad \|u\|_{\omega,r,\varphi} = \sum_{k=0}^{r} \left\| u^{(k)}\varphi^k \right\|_{\omega}$$

are equivalent on $\mathbf{L}^{2,r}_{\omega}$.

For an integer $r \geq 0$, let $\mathbf{C}^r_{\varphi}$ denote the space of all r times differentiable functions $u : (-1, 1) \longrightarrow \mathbb{C}$ satisfying the condition $u^{(k)}\varphi^k \in \mathbf{C}[-1, 1]$ for $k = 0, 1, \ldots, r$.

Corollary 8.2. *(See [20], Remark 1.5, Lemma 3.5.) Let $r \geq 0$ be an integer and let $a \in \mathbf{C}^r_{\varphi}$. Then $a\mathcal{I} \in \mathcal{L}(\mathbf{L}^{2,s}_{\omega})$ for $0 \leq s \leq r$.*

Remark 8.3. Since $\|u\|_{\mu,s,\sim} = \|\mu u\|_{\nu,s}$ and $\|au\|_{\mu,s,\sim} = \|a\mu u\|_{\nu,s}$ we have that $a\mathcal{I} \in \mathcal{L}(\widetilde{\mathbf{L}}^{2,s}_{\mu})$ if and only if $a\mathcal{I} \in \mathcal{L}(\mathbf{L}^{2,s}_{\nu})$.

Regarding the following lemma we refer to [18, Theorem 3.4], [21, Theorem 2.3], and [22, Theorem 3.4].

Lemma 8.4. *Let $\gamma, \delta > -1$ and let $\chi(x) = v^{\gamma,\delta}(x)$ be a Jacobi weight function. For $s > \frac{1}{2}$ and $f \in \mathbf{L}^{2,s}_{\omega}$, the function $f : (-1, 1) \longrightarrow \mathbb{C}$ is continuous. Moreover, we have*

$$\lim_{n\to\infty} \|f - \mathcal{L}^{\chi}_n f\|_{\omega,s} = 0 \quad \text{and} \quad \|f - \mathcal{L}^{\chi}_n f\|_{\omega,t} \leq c\, n^{t-s}\, \|f\|_{\omega,s}\,, \quad 0 \leq t \leq s\,,$$

if

$$\frac{\omega}{\chi\,\varphi}, \frac{\chi\,\varphi}{\omega} \in \mathbf{L}^1(-1, 1) \quad \text{i.e.,} \quad \alpha - \frac{3}{2} < \gamma < \alpha + \frac{1}{2} \text{ and } \beta - \frac{3}{2} < \delta < \beta + \frac{1}{2}\,. \tag{8.5}$$

The constant does not depend on f, n, s, t. Moreover, $\mathcal{L}^{\chi}_n$ is the Lagrange interpolation operator w.r.t. the zeros of $p^{\chi}_n(x)$.

Corollary 8.5. *For $s > \frac{1}{2}$ and $f \in \widetilde{\mathbf{L}}^{2,s}_{\mu}$, the function $f : (-1, 1) \longrightarrow \mathbb{C}$ is continuous and*

$$\lim_{n\to\infty} \left\| f - \widehat{\mathcal{M}}^{\sigma}_n f \right\|_{\mu,s,\sim} = 0 \quad \text{and} \quad \left\| f - \widehat{\mathcal{M}}^{\sigma}_n f \right\|_{\mu,t,\sim} \leq c\, n^{t-s}\, \|f\|_{\mu,s,\sim}\,,$$

$0 \leq t \leq s$, where the constant does not depend on f, n, s, t.

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Proof. Since the condition (8.5) is satisfied for $\omega = \nu$ and $\chi = \sigma$, we can conclude

$$\begin{aligned}\left\| f - \widehat{\mathcal{M}}_n^\sigma f \right\|_{\mu,t,\sim} &= \|\nu(\mu f - \mathcal{L}_n^\sigma \mu f)\|_{\mu,t,\sim} = \|\mu f - \mathcal{L}_n^\sigma \mu f\|_{\nu,t} \\ &\le c\, n^{t-s} \|\mu f\|_{\nu,s} = c\, n^{t-s} \|f\|_{\mu,s,\sim} \,,\end{aligned}$$

and the corollary is proved. □

Let $\psi(x) = x$ and $f : (-1,1) \longrightarrow \mathbb{C}$ be an arbitrary function. Using the algebraic accuracy of the Gaussian rule w.r.t. the weight σ, we obtain

$$\begin{aligned}\left\| \widehat{\mathcal{M}}_n^\sigma f \right\|_\mu^2 &= \langle (1+\psi)\mathcal{L}_n^\sigma \mu f, \mathcal{L}_n^\sigma \mu f\rangle_\sigma = \frac{\pi}{n}\sum_{k=1}^n (1 - x_{kn}^\sigma)|f(x_{kn}^\sigma)|^2 \\ &= \langle (1-\psi)\mathcal{L}_n^\sigma f, \mathcal{L}_n^\sigma f\rangle_\sigma = \langle \mathcal{L}_n^\sigma f, \mathcal{L}_n^\sigma f\rangle_\mu = \|\mathcal{L}_n^\sigma f\|_\mu^2 \end{aligned} \tag{8.6}$$

An assertion, analogous to the following one, was already proved in [23, Lemma 6].

Lemma 8.6. *The operators $\widehat{\mathcal{M}}_n^\sigma : \mathbf{BC}_{0,\frac14} \longrightarrow \mathbf{L}_\mu^2$ are bounded, where*

$$\left\| \widehat{\mathcal{M}}_n^\sigma \right\|_{\mathcal{L}\left(\mathbf{BC}_{0,\frac14}, \mathbf{L}_\mu^2\right)} \le c\sqrt{\ln n}\,,$$

where the constant does not depend on n, and where $\mathbf{BC}_{0,\frac14}$ is defined in Remark 4.5.

Proof. If $f \in \mathbf{BC}_{0,\frac14}$, then (cf. (8.6))

$$\begin{aligned}\left\| \widehat{\mathcal{M}}_n^\sigma f \right\|_\mu^2 &\le \|f\|_{0,\frac14,\infty}^2 \frac{2\pi}{n}\sum_{k=1}^n \frac{1}{\sqrt{1 + x_{kn}^\sigma}} \\ &= \|f\|_{0,\frac14,\infty}^2 \frac{\sqrt{2}\pi}{n}\sum_{k=1}^n \frac{1}{cs\,\frac{2k-1}{4n}\pi} \le c\,\|f\|_{0,\frac14,\infty}^2 \ln n\,,\end{aligned}$$

where the constant does not depend on n and f. □

Proposition 8.7. *If the collocation method (8.1) is stable in $\mathbf{L}_\mu^2$, if $u_n^* \in \operatorname{im}\widehat{\mathcal{L}}_n$ is the unique solution of (8.1), if $a \in \mathbf{C}_\varphi^1$, and if the unique solution $u^* \in \mathbf{L}_\mu^2$ of the original equation (2.2) belongs to $\widetilde{\mathbf{L}}_\mu^{2,s}$ for some $s > \frac12$, then*

$$\|u^* - u_n^*\|_{\mu,t,\sim} \le c\, n^{t-s}\sqrt{\ln n}\,\|u^*\|_{\mu,s,\sim}\,, \quad 0 \le t < s\,, \tag{8.7}$$

where the constant is independent of u^*, n, t, s.

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Proof. Using (8.2) and the stability of the method we get

$$\left\|\widehat{\mathcal{L}}_n u^* - u_n^*\right\|_{\mu,t,\sim} \le c\, n^t \left\|\widehat{\mathcal{M}}_n^\sigma \mathcal{A}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu ,$$

such that, in view of (8.3), it remains to show that

$$\left\|\widehat{\mathcal{M}}_n^\sigma \mathcal{A}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu \le c\, n^{-s}\sqrt{\ln n}\, \|u^*\|_{\mu,s,\sim} , \tag{8.8}$$

where $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \mathcal{B}$ with $\mathcal{B} = \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^-$. With the help of Corollary 8.2, Remark 8.3, and Corollary 8.5 we conclude, for $\frac{1}{2} < t_0 < \min\{s, 1\}$,

$$\begin{aligned}\left\|\widehat{\mathcal{M}}_n^\sigma a(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu &\le \left\|(\widehat{\mathcal{M}}_n^\sigma - \mathcal{I}) a(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu + \|a\|_\infty \left\|\widehat{\mathcal{L}}_n u^* - u^*\right\|_\mu \\ &\le c\left(n^{-t_0}\left\|a(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_{\mu,t_0,\sim} + n^{-s}\|u^*\|_{\mu,s,\sim}\right) \\ &\le c\, n^{-s}\|u^*\|_{\mu,s,\sim} .\end{aligned}$$

Since $\widehat{\mathcal{M}}_n^\sigma b\mathcal{S}\widehat{\mathcal{L}}_n = \widehat{\mathcal{M}}_n^\sigma b\widehat{\mathcal{L}}_n \widehat{\mathcal{M}}_n^\sigma \mathcal{S}\widehat{\mathcal{L}}_n$, we can estimate

$$\begin{aligned}\left\|\widehat{\mathcal{M}}_n^\sigma b\mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu &\le c\left\|\widehat{\mathcal{M}}_n^\sigma \mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu \le c\left\|\mathcal{L}_n^\sigma \mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu \\ &\le c\left(\left\|(\mathcal{L}_n^\sigma - \mathcal{I})\mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu + \left\|\mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu\right) \\ &\le c\left(n^{-s}\left\|\widehat{\mathcal{L}}_n u^* - u^*\right\|_{\mu,s,\sim} + \left\|\widehat{\mathcal{L}}_n u^* - u^*\right\|_\mu\right) \\ &\le c\, n^{-s}\|u^*\|_{\mu,s,\sim} ,\end{aligned}$$

where we took into account (8.6), (8.3), (8.4), and Lemma 8.4. Finally, applying Lemma 8.6 and Remark 4.5 together with (8.3) we get

$$\begin{aligned}\left\|\widehat{\mathcal{M}}_n^\sigma \mathcal{B}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu &\le c\sqrt{\ln n}\left\|\mathcal{B}(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_{0,\frac{1}{4},\infty} \\ &\le c\sqrt{\ln n}\left\|\widehat{\mathcal{L}}_n u^* - u^*\right\|_\mu \le c\, n^{-s}\sqrt{\ln n}\,\|u^*\|_{\mu,s,\sim} ,\end{aligned}$$

and the proposition is proved. □

Now, we start to prepare the proof of an error estimate for the collocation–quadrature method. Assume $\alpha, \beta > -1$. Let $v^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ be a Jacobi weight. By

$\mathcal{S}_{\alpha,\beta} : \mathbf{L}^2_{v^{\alpha,\beta}} \longrightarrow \mathbf{L}^2_{v^{\alpha,\beta}}$ we denote the Fourier projections

$$\mathcal{S}_n^{\alpha,\beta} u = \sum_{j=0}^{n-1} \left\langle u, p_j^{\alpha,\beta} \right\rangle_{v^{\alpha,\beta}} p_j^{\alpha,\beta} ,$$

where $p_n^{\alpha,\beta}$ is the n-th normalized Jacobi polynomial w.r.t. to the weight $v^{\alpha,\beta}$ on the interval $[-1, 1]$. For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\gamma, \delta > -\frac{1}{p}$, we define the weighted spaces $\mathbf{L}^p_{\gamma,\delta}$ by

$$\mathbf{L}^p_{\gamma,\delta} = \left\{ f : v^{\gamma,\delta} f \in \mathbf{L}^p(-1, 1) \right\}$$

and the norm $\|f\|_{\mathbf{L}^p_{\gamma,\delta}} = \|f\|_{p,\gamma,\delta} := \left\|v^{\gamma,\delta} f\right\|_p$, where

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \, dx \right)^{\frac{1}{p}} .$$

Lemma 8.8. *(See [24], Theorem 5.1, [25], p. 328.) The sequence of the norms* $\left\|S_n^{\alpha,\beta}\right\|_{\mathbf{L}^p_{\gamma,\delta} \to \mathbf{L}^p_{\gamma,\delta}}$ *is uniformly bounded (w.r.t. $n \in \mathbb{N}$) if and only if*

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} < \gamma + \frac{1}{p} < \min\left\{\alpha + 1, \frac{\alpha}{2} + \frac{3}{4}\right\}$$

and

$$\max\left\{0, \frac{\beta}{2} + \frac{1}{4}\right\} < \delta + \frac{1}{p} < \min\left\{\beta + 1, \frac{\beta}{2} + \frac{3}{4}\right\} .$$

Lemma 8.9. *(See [22], p. 264.) If $1 < p < \infty$, $-\frac{1}{p} < \gamma, \delta < \frac{1}{q}$, and if $f : (-1, 1) \longrightarrow \mathbb{C}$ is locally absolutely continuous, then*

$$\|f - \mathcal{L}_n^\sigma f\|_{p,\gamma,\delta} \le \frac{c}{n} \left(\int_{-1+\frac{1}{n^2}}^{1-\frac{1}{n^2}} |f'(x)|^p (1 - x)^{(\frac{1}{2}+\gamma)p} (1 + x)^{(\frac{1}{2}+\delta)p} \, dx \right)^{\frac{1}{p}} ,$$

where the constant does not depend on f and n.

Proposition 8.10. *Let $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \mathcal{B}$ with $a \in \mathbf{C}^1_\varphi$, $b \in \mathbf{PC}$, and $\mathcal{B} = \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^-$, $\beta_k^- \in \mathbb{C}$. If the collocation–quadrature method (7.3) is stable in $\mathbf{L}^2_\mu$, if $u_n^* \in \operatorname{im} \widehat{\mathcal{L}}_n$ is the unique solution of (7.3), and if the unique solution $u^* \in \mathbf{L}^2_\mu$ of the original equation (2.2)* *belongs to* $\widetilde{\mathbf{L}}^{2,s}_\mu \cap \mathbf{L}^p_{\gamma,\delta}$ *for some* $s > \frac{1}{2}$, $1 < p < \infty$, $\frac{1}{2} - \frac{1}{p} < \gamma < \frac{1}{q}$, *and* $-\frac{1}{p} < \delta < \frac{1}{4} - \frac{1}{p}$, *then*

$$\|u^* - u^*_n\|_{\mu,t,\sim} \le c\, n^{t-2\left(\frac{1}{4}-\frac{1}{p}-\delta\right)} \sqrt{\ln n}\left(\|u^*\|_{\mu,s,\sim} + \|u^*\|_{p,\gamma,\delta}\right), \tag{8.9}$$

$0 \le t < 2\left(\frac{1}{4} - \frac{1}{p} - \delta\right)$, *where the constant is independent of* u^*, n, t, s.

Proof. Set $\mathcal{H}_n = \sum_{k=1}^{m_-} \beta^-_k \mathcal{H}^{-,\sigma}_{n,k}$. As in the proof of Proposition 8.7 we get, using (8.2),

$$\begin{aligned}
&\left\|\widehat{\mathcal{L}}_n u^* - u^*_n\right\|_{\mu,t,\sim} \\
&\quad \le c\, n^t \left\|\widehat{\mathcal{M}}^\sigma_n (a\mathcal{I} + b\mathcal{S} + \mathcal{H}_n)\widehat{\mathcal{L}}_n u^* - \widehat{\mathcal{M}}^\sigma_n f\right\|_\mu \\
&\quad \le c\, n^t \left(\left\|\widehat{\mathcal{M}}^\sigma_n (a\mathcal{I} + b\mathcal{S} + \mathcal{B})(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu + \left\|\widehat{\mathcal{M}}^\sigma_n (\mathcal{H}_n - \mathcal{B})\widehat{\mathcal{L}}_n u^*\right\|_\mu\right).
\end{aligned}$$

Taking into account (8.8) and $2\left(\frac{1}{4} - \frac{1}{p} - \delta\right) < \frac{1}{2} < s$, we observe that it suffices to prove the estimate

$$\left\|\widehat{\mathcal{M}}^\sigma_n (\mathcal{H}_n - \mathcal{B})\widehat{\mathcal{L}}_n u^*\right\|_\mu \le c\, n^{2\delta + \frac{2}{p} - \frac{1}{2}} \sqrt{\ln n}\, \|\mu u^*\|_{p,\gamma,\delta}. \tag{8.10}$$

Define $h(x,y) = \dfrac{1}{\pi \mathbf{i}} \displaystyle\sum_{k=1}^{m_-} \frac{\beta^-_k (1+x)^{k-1}}{(2+x+y)^k}$ and $H_x(y) = (1+x)^{\frac{1}{4}} h(x,y)$. Then, for $-1 < x, y < 1$,

$$|H'_x(y)| \le c\,(1+y)^{-\frac{7}{4}}, \tag{8.11}$$

where the constant does not depend on x and y. By $\mathcal{L}^\sigma_{ny}$ we refer to the application of the Lagrange interpolation operator $\mathcal{L}^\sigma_n$ with respect to the variable $y \in (-1,1)$. If $u_n = \nu p_n$ with a polynomial p_n of degree less than n, then

$$\begin{aligned}
\int_{-1}^{1} u_n(y)\left(\mathcal{L}^\sigma_{ny} h(x,\cdot)\right)(y)\, dy &= \int_{-1}^{1} (1+y) p_n(y) \left(\mathcal{L}^\sigma_{ny} h(x,\cdot)\right)(y) \frac{dy}{\sqrt{1-y^2}} \\
&= \frac{\pi}{n} \sum_{k=1}^{n} (1 + x^\sigma_{kn}) p_n(x^\sigma_{kn}) h(x, x^\sigma_{kn}) \\
&= \frac{\pi}{n} \sum_{k=1}^{n} \varphi(x^\sigma_{kn}) h(x, x^\sigma_{kn}) u_n(x^\sigma_{kn}) = \left(\mathcal{H}_n u_n\right)(x),
\end{aligned}$$

$-1 < x < 1$. Consequently, for $-1 < x < 1$,

$$\left((\mathcal{H}_n - \mathcal{B})\widehat{\mathcal{L}}_n u^*\right)(x) = \int\limits_{-1}^{1} \left[\left(\mathcal{L}^\sigma_{ny} h(x,\cdot)\right)(y) - h(x,y)\right] \left(\widehat{\mathcal{L}}_n u^*\right)(y)\, dy\,. \tag{8.12}$$

From $\widehat{\mathcal{L}}_n f = \nu \sum\limits_{j=0}^{n-1} \langle f, \nu R_n\rangle_\mu R_n = \nu \sum\limits_{j=0}^{n-1} \langle \mu f, R_n\rangle_\nu R_n$ we get the relation $\widehat{\mathcal{L}}_n f = \nu \mathcal{S}_n^{-\frac12,\frac12} \mu f$. In view of Lemma 8.8 we have $\left\|\nu \mathcal{S}_n^{-\frac12,\frac12}\mu\mathcal{I}\right\|_{\mathbf{L}^p_{\gamma,\delta}\to\mathbf{L}^p_{\gamma,\delta}} \le c$ if $0 < \gamma - \frac12 + \frac1p < \frac12$ and $\frac12 < \delta + \frac12 + \frac1p < 1$, i.e., if $\frac12 - \frac1p < \gamma < \frac1q$ and $-\frac1p < \delta < \frac12 - \frac1p$. Now, with the help of Lemma 8.6 and (8.12), we can estimate

$$\begin{aligned}
&\left\|\widehat{\mathcal{M}}^\sigma_n (\mathcal{H}_n - \mathcal{B})\widehat{\mathcal{L}}_n u^*\right\|_\mu \\
&\le c\sqrt{\ln n} \sup_{-1<x<1} \left| \int\limits_{-1}^{1} \left[\left(\mathcal{L}^\sigma_n H_x\right)(y) - H_x(y)\right] \left(\widehat{\mathcal{L}}_n u^*\right)(y)\, dy \right| \\
&\le c\sqrt{\ln n} \sup_{-1<x<1} \left\|\mathcal{L}^\sigma_n H_x - H_x\right\|_{q,-\gamma,-\delta} \left\|\nu \mathcal{S}_n^{-\frac12,\frac12}\mu u^*\right\|_{p,\gamma,\delta} \\
&\le c\sqrt{\ln n} \sup_{-1<x<1} \left\|\mathcal{L}^\sigma_n H_x - H_x\right\|_{q,-\gamma,-\delta} \left\|u^*\right\|_{p,\gamma,\delta}\,.
\end{aligned}$$

By Lemma 8.9 and (8.11) we get

$$\left\|\mathcal{L}^\sigma_n H_x - H_x\right\|_{q,-\gamma,-\delta} \le \frac{c}{n} \left(1 + \int\limits_{-1+\frac{1}{n^2}}^{0} (1+y)^{-\left(\frac54+\delta\right)q}\, dy\right)^{\frac1q} = c\, n^{2\delta + \frac2p - \frac12}\,,$$

and (8.10) is proved. □

Let $\gamma, \delta \ge 0$. By $E^{\gamma,\delta}_n(f)$ we denote the best weighted uniform approximation of $f \in \mathbf{C}_{\gamma,\delta}$ by polynomials of degree less than n, i.e.,

$$E^{\gamma,\delta}_n(f) := \inf\left\{\|f - p\|_{\gamma,\delta,\infty} : p \in \mathbb{P}_n\right\}, \quad n > 0\,.$$

Moreover, let $E^{\gamma,\delta}_0(f) := \|f\|_{\gamma,\delta,\infty}$. For $s > 0$, we define the subspace $\mathbf{C}^s_{\gamma,\delta}$ of $\mathbf{C}_{\gamma,\delta}$ by (cf. [26, (2.9)])

$$\mathbf{C}^s_{\gamma,\delta} = \left\{ f \in \mathbf{C}_{\gamma,\delta} : \sup_{n\ge 0} (n+1)^s E^{\gamma,\delta}_n(f) < \infty \right\}$$

and equip it with the norm

$$\|f\|_{\mathbf{C}^s_{\gamma,\delta}} := \sup_{n\geq 0}(n+1)^s E_n^{\gamma,\delta}(f)\,.$$

In the spaces $\mathbf{C}_{\gamma,\delta}$, instead of Lemma 8.8 the following is true.

Lemma 8.11. *(See [25], Theorem 2.1.) If $\gamma < 1+\alpha$, $\delta < 1+\beta$, and if*

$$\max\left\{0, \frac{\alpha}{2}+\frac{1}{4}\right\} \leq \gamma \leq \frac{\alpha}{2}+\frac{3}{4}, \quad \max\left\{0, \frac{\beta}{2}+\frac{1}{4}\right\} \leq \delta \leq \frac{\beta}{2}+\frac{3}{4},$$

then

$$\left\|\mathcal{S}_n^{\alpha,\beta}\right\|_{\mathbf{C}_{\gamma,\delta}\to\mathbf{C}_{\gamma,\delta}} \leq c\,\ln n\,,$$

where the constant does not depend on n.

Lemma 8.12. *(See [26], Theorem 3.3.) For all $s > 0$, the linear operator $\mathcal{S}\nu\mathcal{I} : \mathbf{C}^s_{0,\frac{1}{2}} \longrightarrow \mathbf{C}^s_{\frac{1}{2},0}$ is bounded.*

Proposition 8.13. *Let $\mathcal{A} = a\mathcal{I} + b\mathcal{S} + \mathcal{B}$ with $a, b \in \mathbf{PC}$ and $\mathcal{B} = \sum_{k=1}^{m_-} \beta_k^- \mathcal{B}_k^-$, $\beta_k^- \in \mathbb{C}$. If the collocation–quadrature method (7.3) is stable in $\mathbf{L}^2_\mu$, if $u_n^* \in \operatorname{im}\widehat{\mathcal{L}}_n$ is the unique solution of (7.3), and if the unique solution $u^* \in \mathbf{L}^2_\mu$ of the original equation (2.2) has the properties $\mu u^* \in \mathbf{C}^s_{0,\frac{1}{2}}$ and $u^* \in \mathbf{L}^p_{\gamma,\delta}$ for some $s > 0$, $1 < p < \infty$, $\frac{1}{2}-\frac{1}{p} < \gamma < \frac{1}{q}$, and $-\frac{1}{p} < \delta < \frac{1}{4}-\frac{1}{p}$, then, for every $\varepsilon \in (0, s)$,*

$$\|u^* - u_n^*\|_\mu \leq c\left[n^{\varepsilon-s}(\ln n)^{\frac{3}{2}}\,\|\mu u^*\|_{\mathbf{C}^s_{0,\frac{1}{2}}} + n^{-2\left(\frac{1}{4}-\frac{1}{p}-\delta\right)}\sqrt{\ln n}\,\|u^*\|_{p,\gamma,\delta}\right], \quad (8.13)$$

where the constant is independent of u^*, n, and s.

Proof. We start as in the proof of Proposition 8.10 and get

$$\begin{aligned}&\left\|\widehat{\mathcal{L}}_n u^* - u^*\right\|_\mu \\ &\quad\leq c\left(\left\|\widehat{\mathcal{M}}_n^\sigma(a\mathcal{I}+b\mathcal{S}+\mathcal{B})(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu + \left\|\widehat{\mathcal{M}}_n^\sigma(\mathcal{H}_n - \mathcal{B})\widehat{\mathcal{L}}_n u^*\right\|_\mu\right).\end{aligned}$$

Having in mind the proof of Proposition 8.10 we have to show that

$$\left\|\widehat{\mathcal{M}}_n^\sigma(a\mathcal{I}+b\mathcal{S}+\mathcal{B})(\widehat{\mathcal{L}}_n u^* - u^*)\right\|_\mu \leq c\,n^{\varepsilon-s}(\ln n)^{\frac{3}{2}}\,\|\mu u^*\|_{0,\frac{1}{2},\infty}\,. \quad (8.14)$$

The relations (8.6) show that, for every function $f : (-1, 1) \longrightarrow \mathbb{C}$,

$$\left\| \widehat{\mathcal{M}}_n^\sigma f \right\|_\mu \le \sqrt{\pi} \sup \left\{ \sqrt{1-x}\, |f(x)| : -1 < x < 1 \right\} . \tag{8.15}$$

Hence,

$$\begin{aligned} \left\| \widehat{\mathcal{M}}_n^\sigma a(\widehat{\mathcal{L}}_n u^* - u^*) \right\|_\mu &\le c \left\| \widehat{\mathcal{L}}_n u^* - u^* \right\|_{\frac{1}{2},0,\infty} \\ &= c \left\| \mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} \mu u^* - \mu u^* \right\|_{0,\frac{1}{2},\infty} . \end{aligned} \tag{8.16}$$

With the help of Lemma 8.12, analogously we get

$$\begin{aligned} \left\| \widehat{\mathcal{M}}_n^\sigma b\mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*) \right\|_\mu &\le c \left\| \mathcal{S}(\widehat{\mathcal{L}}_n u^* - u^*) \right\|_{\frac{1}{2},0,\infty} \\ &\le c \left\| \mathcal{S}\nu \left(\mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} \mu u^* - \mu u^* \right) \right\|_{\mathbf{C}^\varepsilon_{\frac{1}{2},0}} \\ &\le c \left\| \mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} \mu u^* - \mu u^* \right\|_{\mathbf{C}^\varepsilon_{0,\frac{1}{2}}} . \end{aligned} \tag{8.17}$$

Since, for $-1 < x < 1$,

$$(1+x)^{\frac{1}{4}} \int_{-1}^{1} \frac{(1+x)^{k-1}\nu(y)|f(y)|}{(2+x+y)^k}\, dy \le \int_{-1}^{1} \frac{dy}{(1-y)^{\frac{1}{2}}(1+y)^{\frac{3}{4}}}\ \|f\|_{0,\frac{1}{2},\infty} ,$$

we observe, using Lemma 8.6, that

$$\begin{aligned} \left\| \widehat{\mathcal{M}}_n^\sigma \mathcal{B}(\widehat{\mathcal{L}}_n u^* - u^*) \right\|_\mu &\le c \sqrt{\ln n} \left\| \mathcal{B}\nu \left(\mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} \mu u^* - \mu u^* \right) \right\|_{0,\frac{1}{4},\infty} \\ &\le c \sqrt{\ln n} \left\| \mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} \mu u^* - \mu u^* \right\|_{0,\frac{1}{2},\infty} . \end{aligned} \tag{8.18}$$

For $f \in \mathbf{C}_{0,\frac{1}{2}}$ and $p \in \mathbb{P}_n$, we have

$$\left\| \mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} f - f \right\|_{0,\frac{1}{2},\infty} \le \left(\left\| \mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} \right\|_{\mathbf{C}_{0,\frac{1}{2}} \to \mathbf{C}_{0,\frac{1}{2}}} + 1 \right) \|f - p\|_{0,\frac{1}{2},\infty} ,$$

such that, due to Lemma 8.11 and the definition of $\mathbf{C}^s_{0,\frac{1}{2}}$,

$$\left\| \mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} f - f \right\|_{0,\frac{1}{2},\infty} \le c \ln n\, E_n^{0,\frac{1}{2}}(f) \le c\, n^{-s} \ln n\ \|f\|_{\mathbf{C}^s_{0,\frac{1}{2}}} \tag{8.19}$$

for $s > 0$ and $f \in \mathbf{C}^s_{0,\frac{1}{2}}$. Moreover, for $0 < \varepsilon < s$,

$$
\begin{aligned}
&\left\|\mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} f - f\right\|_{\mathbf{C}^{\varepsilon}_{0,\frac{1}{2}}} \\
&\quad \le \max\left\{\max_{k=0,\ldots,n}(k+1)^{\varepsilon} E_k^{0,\frac{1}{2}}\left(\mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} f - f\right), \sup_{k>n}(k+1)^{\varepsilon} E_k^{0,\frac{1}{2}}(f)\right\} \\
&\quad \le \max\left\{(n+1)^{\varepsilon}\left\|\mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} f - f\right\|_{0,\frac{1}{2},\infty}, \sup_{k>n}(k+1)^{\varepsilon}(k+1)^{-s}\|f\|_{\mathbf{C}^{s}_{0,\frac{1}{2}}}\right\}
\end{aligned}
$$

and, taking into account (8.19),

$$
\left\|\mathcal{S}_n^{-\frac{1}{2},\frac{1}{2}} f - f\right\|_{\mathbf{C}^{\varepsilon}_{0,\frac{1}{2}}} \le c\, n^{\varepsilon-s} \ln n \, \|f\|_{\mathbf{C}^{\varepsilon}_{0,\frac{1}{2}}} . \tag{8.20}
$$

Putting (8.16), (8.17), (8.18), and (8.19), (8.20) together, the desired estimate (8.14) follows. □

Example 8.14. Let us consider equation (7.1) for the right hand side

$$
f(x) = \frac{2x}{\pi}\sqrt{\frac{1+x}{1-x}} \ln \frac{1+\sqrt{1-x^2}}{|x|} - g(x) - 6(1+x)g'(x) - 2(1+x)g''(x) \tag{8.21}
$$

with

$$
g(x) = (x+2)\sqrt{\frac{1+x}{3+x}}\left(\frac{4}{\pi}\arctan\sqrt{\frac{1+x}{3+x}} - 1\right) .
$$

Then, the solution of this equation is $u^*(x) = |x|\sqrt{\dfrac{1+x}{1-x}}$. We get, for $n = 0, 1, 2, \ldots$,

$$
\langle u^*, \nu R_n\rangle_{\mu} = \int_{-1}^{1} |x|\, R_n(x)\sqrt{\frac{1+x}{1-x}}\, dx = \frac{2}{\sqrt{\pi}}\begin{cases} \dfrac{(-1)^{\frac{n}{2}+1}}{n^2-1} : & n \text{ even}, \\[2ex] \dfrac{(-1)^{\frac{n-1}{2}}}{n(n+2)} : & n \text{ odd}, \end{cases}
$$

which implies that $u^* \in \widetilde{\mathbf{L}}^{2,s}_{\mu}$ for $s < \frac{3}{2}$. Since from Corollary 8.5 we know that

$$
\left\|u^* - \widehat{\mathcal{M}}_n^{\sigma} u^*\right\|_{\mu} \le c\, n^{-s}\, \|u^*\|_{\mu,s,\sim}
$$

holds, we compute, using the numerical solution $u_n^* = \nu p_n^*$ of the collocation or collocation–quadrature method with p_n^* being a polynomial of degree less than n,

$$
\begin{aligned}
\left\|\widehat{\mathcal{M}}_n^{\sigma} u^* - u_n^*\right\|_{\mu} &= \|\mathcal{L}_n^{\sigma}\mu u^* - p_n^*\|_{\nu} \\
&= \sqrt{\int_{-1}^{1} (1+x)\left|\left(\mathcal{L}_n^{\sigma}\mu u^*\right)(x) - p_n^*(x)\right|^2 \frac{dx}{\sqrt{1-x^2}}}
\end{aligned}
$$

Table 8
The values $n^{\frac{3}{2}} \left\| \widehat{\mathcal{M}}_n^\sigma u^* - u_n^* \right\|_\mu$ for (7.1) with $f(x)$ given in (8.21) and for $\tau = \sigma$.

n	CM (8.1)	CQM (7.3)
16	1.097	1.1846
32	1.086	1.1557
64	1.083	1.1446
128	1.082	1.1399
256	1.082	1.1377
512	1.082	1.1367
1024	1.082	1.1362
2048	1.082	1.1360
4096	1.082	1.1358
8192	1.082	1.1358
16384	1.082	1.1357

$$= \sqrt{\frac{\pi}{n} \sum_{k=1}^{n} (1 + x_{kn}^\sigma) \left| \mu(x_{kn}^\sigma) u^*(x_{kn}^\sigma) - p_n^*(x_{kn}^\sigma) \right|^2}$$

$$= \sqrt{\frac{\pi}{n} \sum_{k=1}^{n} (1 - x_{kn}^\sigma) \left| u^*(x_{kn}^\sigma) - u_n^*(x_{kn}^\sigma) \right|^2} \,.$$

In the following table we present the values $n^{\frac{3}{2}} \left\| \widehat{\mathcal{M}}_n^\sigma u^* - u_n^* \right\|_\mu$ for the collocation and for the collocation-quadrature method w.r.t. the nodes x_{kn}^σ applied to equation (7.1) with the right-hand side (8.21). The results verify Proposition 8.7 for the collocation method. In Propositions 8.10 and 8.13 for the collocation–quadrature method one can only state convergence rates of order less than $\frac{1}{2}$. Hence, the results given in Table 8 for the collocation–quadrature method show that the orders of convergence presented in these two propositions seem to be improvable.

In case $f(x)$ is constant, as described at the beginning of Section 7, from [4, Theorem 14.1] we infer that the solution $u(x) = u^*(x)$ of equation (7.1) is infinitely differentiable in $(-1, 1)$ and that $\sqrt{1-x}\, u^*(x)$ is bounded and Hölder continuous on $(-1, 1]$. Unfortunately, we cannot conclude from these regularity results the conditions assumed for u^* in the propositions of this section. For example, concerning Proposition 8.13, we get $\mu u^* \in \mathbf{BC}_{0,\frac{1}{2}}$, but not $\mu u^* \in \mathbf{C}_{0,\frac{1}{2}}$.

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