

A Proof of the Marcinkiewicz-Zygmund Inequality on the Bi-Sphere

Frank Filbir and Daniel Potts

ABSTRACT. In this technical report we present a complete proof of the Marcinkiewicz-Zygmund inequality on the bi-Sphere from [1, Theorem 4.5].

We use the same notation as in [1]. The Theorem 4.5 of [1] reads as follows:

Theorem 1. *Let $N \in \mathbb{N}^2$ and \mathcal{R} be a admissible decomposition of $\mathbb{S}^2 \times \mathbb{S}^2$ according to a sampling set \mathcal{X} such that*

$$\|\mathcal{R}\| \leq \frac{\eta}{21\tilde{N}}, \quad (1)$$

where $\eta \in (0, 1)$ is arbitrarily fixed. Then for $r = 1$ or $r = \infty$ and any $p \in \Pi_N$, we have

$$(1 - \eta) \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, r} \leq \|p\|_{\mathcal{X}, r} \leq (1 + \eta) \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, r}. \quad (2)$$

We apply subsequently the following inequalities:

Lemma 2. *The following estimates are valid,*

- (i) $\max_{t \in [0, \pi]} |v_N(\cos t)| \leq \frac{1}{\pi} N^2,$
- (ii) $\int_{\xi_j^-}^{\xi_j^+} \left| \frac{d}{dt} v_N(\cos t) \right| dt \leq \frac{2}{\pi} N^3 \|\mathcal{R}\|,$
- (iii) $\int_{\|\mathcal{R}\|}^{\pi - \|\mathcal{R}\|} \left| \frac{d}{dt} v_N(\cos t) \right| \sin t dt \leq \frac{3}{2\pi} (5N + 1).$

Proof of Lemma 2: From [2, equations (3.13), (3.14)], we obtain immediately the inequality (i). Using the Bernstein inequality $\|T'\|_\infty \leq 2N\|T\|_\infty$ for $v_N(\cos t)$ and (i) we get (ii). The integral in (iii) can be estimated as in the proof of Theorem 4.2 in [2]. ■

Proof of Theorem 1: We start with the case $r = \infty$. For proving the left-hand side let $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^2 \times \mathbb{S}^2$ an arbitrary fixed point and assume that $(\mathbf{u}, \mathbf{v}) \in \mathcal{X}$ is a point with $d((\mathbf{x}, \mathbf{y}), (\mathbf{u}, \mathbf{v})) = d((\mathbf{x}, \mathbf{y}), \mathcal{X}) = \delta_{\mathcal{X}}$. We conclude by [1, Lemma 4.4]

$$\begin{aligned} |p(\mathbf{x}, \mathbf{y})| &\leq |p(\mathbf{x}, \mathbf{y}) - p(\mathbf{u}, \mathbf{v})| + |p(\mathbf{u}, \mathbf{v})| \\ &\leq 2\tilde{N} d((\mathbf{x}, \mathbf{y}), (\mathbf{u}, \mathbf{v})) \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, \infty} + \max_{j=1, \dots, M} |p(\mathbf{x}_j, \mathbf{y}_j)|. \end{aligned}$$

Taking the maximum over $(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^2 \times \mathbb{S}^2$ we obtain by using the fact $d((\mathbf{x}, \mathbf{y}), (\mathbf{u}, \mathbf{v})) = d((\mathbf{x}, \mathbf{y}), \mathcal{X}) \leq \|\mathcal{R}\|$ the L^∞ -MZ inequality

$$\left(1 - 2\tilde{N}\|\mathcal{R}\|\right) \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, \infty} \leq \max_{j=1, \dots, M} |p(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)| \leq \left(1 + 2\tilde{N}\|\mathcal{R}\|\right) \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, \infty}. \quad (3)$$

This provides in combination with (1) the inequality (2) for $r = \infty$. For $r = 1$ we are going to show both inequalities of (2) simultaneously by proving

$$\left| \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, 1} - \|p\|_{\mathcal{X}, 1} \right| \leq \eta \|p\|_{\mathbb{S}^2 \times \mathbb{S}^2, 1}.$$

Using [1, Theorem 4.3 (i)] we obtain

$$\begin{aligned} & \left| \int_{\mathbb{S}^2 \times \mathbb{S}^2} |p(\mathbf{x}, \mathbf{y})| \, d\tau(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^M |p(\mathbf{x}_j, \mathbf{y}_j)| \tau(R_j) \right| \\ &= \left| \sum_{j=1}^M \int_{R_j} |p(\mathbf{x}, \mathbf{y})| - |p(\mathbf{x}_j, \mathbf{y}_j)| \, d\tau(\mathbf{x}, \mathbf{y}) \right| \\ &\leq \sum_{j=1}^M \int_{R_j} |p(\mathbf{x}, \mathbf{y}) - p(\mathbf{x}_j, \mathbf{y}_j)| \, d\tau(\mathbf{x}, \mathbf{y}) \\ &\leq \sum_{j=1}^M \int_{R_j} |\mathcal{V}_N p(\mathbf{x}, \mathbf{y}) - \mathcal{V}_N p(\mathbf{x}_j, \mathbf{y}_j)| \, d\tau(\mathbf{x}, \mathbf{y}) \\ &\leq \sup_{(\mathbf{u}, \mathbf{v}) \in \mathbb{S}^2 \times \mathbb{S}^2} \left(\sum_{j=1}^M \int_{R_j} |v_N(\mathbf{x} \cdot \mathbf{u}, \mathbf{y} \cdot \mathbf{v}) - v_N(\mathbf{x}_j \cdot \mathbf{u}, \mathbf{y}_j \cdot \mathbf{v})| \, d\tau(\mathbf{x}, \mathbf{y}) \right) \\ &\quad \times \int_{\mathbb{S}^2 \times \mathbb{S}^2} |p(\mathbf{u}, \mathbf{v})| \, d\tau(\mathbf{u}, \mathbf{v}). \end{aligned}$$

It suffices to show

$$\sup_{(\mathbf{u}, \mathbf{v}) \in \mathbb{S}^2 \times \mathbb{S}^2} \left(\sum_{j=1}^M \int_{R_j} |v_N(\mathbf{x} \cdot \mathbf{u}, \mathbf{y} \cdot \mathbf{v}) - v_N(\mathbf{x}_j \cdot \mathbf{u}, \mathbf{y}_j \cdot \mathbf{v})| \, d\tau(\mathbf{x}, \mathbf{y}) \right) \leq \eta. \quad (4)$$

To this end we use spherical coordinates for the spheres with respect to the point $(\mathbf{u}, \mathbf{v}) \in$

$\mathbb{S}^2 \times \mathbb{S}^2$. This leads to $\mathbf{x} \cdot \mathbf{u} = \cos \xi$, $\mathbf{y} \cdot \mathbf{v} = \cos \eta$. We obtain

$$\begin{aligned} \Sigma &:= \sum_{j=1}^M \int_{R_j} |v_{\mathbf{N}}(\mathbf{x} \cdot \mathbf{u}, \mathbf{y} \cdot \mathbf{v}) - v_{\mathbf{N}}(\mathbf{x}_j \cdot \mathbf{u}, \mathbf{y}_j \cdot \mathbf{v})| \, d\tau(\mathbf{x}, \mathbf{y}) \\ &= \sum_{j=1}^M \int_{R_j} |v_{N_1}(\cos \xi) v_{N_2}(\cos \eta) - v_{N_1}(\cos \xi_j) v_{N_2}(\cos \eta_j)| \, d\tau(\mathbf{x}, \mathbf{y}) \\ &= \sum_{j=1}^M \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \, dt \, d\tau(\mathbf{x}, \mathbf{y}) \\ &\quad + \sum_{j=1}^M \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \, dt \, d\tau(\mathbf{x}, \mathbf{y}). \end{aligned}$$

We denote the first sum in the expression above by Σ_1 and the second sum by Σ_2 . For further estimation of the above expression we introduce a partition of $\mathbb{S}^2 \times \mathbb{S}^2$ in the following way. Let $L = \lfloor \frac{\pi}{\|\mathcal{R}\|} \rfloor$ and define $\mathcal{I}_\nu = [(\nu - 1)\pi L^{-1}, (\nu + 1)\pi L^{-1}]$, $\nu = 1, \dots, L - 1$. For $\ell_1, \ell_2 \in \{1, \dots, L - 1\}$ let

$$B_{\ell_1, \ell_2} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{S}^2 \times \mathbb{S}^2 : (\xi, \eta) \in \mathcal{I}_{\ell_1} \times \mathcal{I}_{\ell_2}\},$$

where $(\mathbf{x}, \mathbf{y}) = (\sin \xi \sin \varphi, \sin \xi \cos \varphi, \cos \xi, \sin \eta \sin \phi, \sin \eta \cos \phi, \cos \eta)$, $\xi, \eta \in [0, \pi]$, $\varphi, \phi \in [-\pi, \pi[$. Let further $B_\nu = \{\mathbf{x} \in \mathbb{S}^2 : \xi \in \mathcal{I}_\nu\}$. Then we have

$$(i) \quad B_{\ell_1, \ell_2} = B_{\ell_1} \times B_{\ell_2}, \text{ where } B_{\ell_\nu} = \{\mathbf{x} \in \mathbb{S}^2; \xi \in \mathcal{I}_\nu\},$$

$$(ii) \quad \bigcup_{\ell_1, \ell_2=1}^{L-1} B_{\ell_1, \ell_2} = \mathbb{S}^2 \times \mathbb{S}^2,$$

$$(iii) \quad B_{\ell_1, \ell_2} \cap B_{\ell'_1, \ell'_2} = \emptyset \text{ for } (\ell_1 > \ell'_1 + 2 \vee \ell_1 < \ell'_1 - 2) \wedge (\ell_2 > \ell'_2 + 2 \vee \ell_2 < \ell'_2 - 2).$$

This provides a decomposition of $\mathbb{S}^2 \times \mathbb{S}^2$ into overlapping regions. Let us denote by ξ_j^-, η_j^- and ξ_j^+, η_j^+ the minimal resp. maximal value of the coordinates ξ resp. η of the region R_j . Since

$$\max\{\xi_j^+ - \xi_j^-, \eta_j^+ - \eta_j^-\} \leq \text{diam}(R_j) \leq \|\mathcal{R}\| \leq \pi L^{-1},$$

the region R_j is contained completely in at least one B_{ℓ_1, ℓ_2} with $\ell_1, \ell_2 \in \{1, \dots, L - 1\}$. Accordingly we have

$$\begin{aligned} \Sigma_1 &\leq \sum_{\ell_1=1}^{L-1} \sum_{\ell_2=1}^{L-1} \sum_{R_j \subset B_{\ell_1, \ell_2}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \, dt \, d\tau(\mathbf{x}, \mathbf{y}), \\ \Sigma_2 &\leq \sum_{\ell_1=1}^{L-1} \sum_{\ell_2=1}^{L-1} \sum_{R_j \subset B_{\ell_1, \ell_2}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \, dt \, d\tau(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Now we have to distinguish between different cases. These cases are

$$\begin{aligned}
(1.1) : \ell_1 = 1 \wedge \{\ell_2 = 1 \vee \ell_2 = L - 1\}, \quad (1.2) : \ell_1 = L - 1 \wedge (\ell_2 = 1 \vee \ell_2 = L - 1), \\
(1.3) : \ell_2 = 1 \wedge (\ell_1 = 1 \vee \ell_1 = L - 1), \quad (1.4) : \ell_2 = L - 1 \wedge (\ell_1 = 1 \vee \ell_1 = L - 1), \\
(2.1) : \ell_1 = 1 \wedge \ell_2 \in \{2, \dots, L - 2\}, \quad (2.2) : \ell_1 = L - 1 \wedge \ell_2 \in \{2, \dots, L - 2\}, \\
(2.3) : \ell_2 = 1 \wedge \ell_1 \in \{2, \dots, L - 2\}, \quad (2.4) : \ell_2 = L - 1 \wedge \ell_1 \in \{2, \dots, L - 2\}, \\
(3) : \ell_1 \in \{2, \dots, L - 2\} \wedge \ell_2 \in \{2, \dots, L - 2\}.
\end{aligned}$$

We now treat the various cases separately.

Case (1.1): Using Lemma 2 (i),(ii) we get

$$\begin{aligned}
S_{1,1}^{(1)} &= \sum_{R_j \subset B_{1,1} \cup B_{1,L-1}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\stackrel{\text{Lemma 2.(ii)}}{\leq} \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \frac{1}{\pi} N_2^2 \{ \tau(B_{1,1}) + \tau(B_{1,L-1}) \} \\
&\leq \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \frac{1}{\pi} N_2^2 \left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \int_0^{2\pi L^{-1}} \right. \\
&\quad \left. + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{\pi-2\pi L^{-1}}^{\pi} \int_{\pi-2\pi L^{-1}}^{\pi} \right\} \sin \xi \sin \eta d\xi d\eta d\varphi d\phi \\
&\leq \frac{2}{\pi^2} N_1^3 \|\mathcal{R}\| N_2^2 4\pi^2 2 \left(\int_0^{2\pi L^{-1}} \sin t dt \right)^2 \\
&\leq 16 N_2^2 N_1^3 \|\mathcal{R}\| (\pi L^{-1})^4 \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 1024 N_2^2 N_1^3 \|\mathcal{R}\|^5.
\end{aligned}$$

It is obvious that due to symmetry we have $S_{1,k}^{(1)} \leq 1024 N_1^3 N_2^2 \|\mathcal{R}\|$ for $k = 2, 3, 4$.

Case (2.1):

$$\begin{aligned}
S_{2,1}^{(1)} &:= \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{1,\ell_2}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_2=2}^{L-2} \int_{B_{1,\ell_2}} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y})
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{B_1} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} |v_{N_2}(\cos \eta)| d\sigma(\mathbf{y}) \\
\text{Lemma 2.(ii)} \quad &\leq \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \sigma(B_1) \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} |v_{N_2}(\cos \eta)| \sin \eta d\eta d\phi \\
&\leq 4N_1^3 \|\mathcal{R}\| \sigma(B_1) \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} |v_{N_2}(\cos \eta)| \sin \eta d\eta d\phi \\
&\leq 8N_1^3 \|\mathcal{R}\| \sigma(B_1) \int_{\pi L^{-1}}^{(L-1)\pi L^{-1}} |v_{N_2}(\cos \eta)| \sin \eta d\eta \\
&\leq 8N_1^3 \|\mathcal{R}\| \sigma(B_1) \int_{\pi L^{-1}}^{\pi} |v_{N_2}(\cos \eta)| \sin \eta d\eta \\
&\leq 8N_1^3 \|\mathcal{R}\| \|v_{N_2}\|_{\infty} \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \sin \xi d\xi d\varphi \int_{\pi L^{-1}}^{\pi} \sin \eta d\eta \\
&\leq 8N_1^3 \|\mathcal{R}\| \frac{N_2^2}{\pi} 2\pi \int_0^{2\pi L^{-1}} \sin \xi d\xi \int_{\pi L^{-1}}^{\pi} \sin \eta d\eta \\
&\leq 16N_1^3 N_2^2 \|\mathcal{R}\| \left[\frac{1}{2} \xi^2 \right]_0^{2\pi L^{-1}} [-\cos \eta]_{\pi L^{-1}}^{\pi} \\
&\leq 32N_1^3 N_2^2 \|\mathcal{R}\| (\pi L^{-1})^2 (1 - \cos(\pi L^{-1})) \\
&\leq 32N_1^3 N_2^2 \|\mathcal{R}\| (\pi L^{-1})^2 2 \sin^2\left(\frac{\pi L^{-1}}{2}\right) \\
&\leq 64N_1^3 N_2^2 \|\mathcal{R}\| (\pi L^{-1})^2 \left(\frac{\pi L^{-1}}{2}\right)^2 \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 256 N_1^3 N_2^2 \|\mathcal{R}\|^5
\end{aligned}$$

Case (2.2):

$$\begin{aligned}
S_{2,2}^{(1)} &:= \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{L-1, \ell_2}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_2=2}^{L-2} \int_{B_{1, \ell_2}} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \int_{B_{L-1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} |v_{N_2}(\cos \eta)| d\sigma(\mathbf{y})
\end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{Lemma 2.(ii)}}{\leq} \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \sigma(B_{L-1}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} |v_{N_2}(\cos \eta)| \, d\sigma(\mathbf{y}) \\
& \leq 256 N_1^3 N_2^2 \|\mathcal{R}\|^5
\end{aligned}$$

Case (2.3):

$$\begin{aligned}
S_{2,3}^{(1)} & := \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1,1}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt \, d\tau(\mathbf{x}, \mathbf{y}) \\
& \leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1,1}} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt \, d\tau(\mathbf{x}, \mathbf{y}) \\
& \leq \int_{B_1} |v_{N_2}(\cos \eta)| \, d\sigma(\mathbf{y}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt \, d\sigma(\mathbf{x}) \\
& \leq \int_{B_1} |v_{N_2}(\cos \eta)| \, d\sigma(\mathbf{y}) \sum_{\ell_1=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt \, \sin \xi \, d\xi \, d\varphi \\
& \stackrel{\sin \xi \leq 2 \sin t}{\leq} 4\pi \int_{B_1} |v_{N_2}(\cos \eta)| \, d\sigma(\mathbf{y}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t \, dt \, d\xi \\
& \leq 4\pi(\pi L^{-1}) \int_{B_1} |v_{N_2}(\cos \eta)| \, d\sigma(\mathbf{y}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t \, dt \\
& \leq 4\pi(\pi L^{-1}) \int_{B_1} |v_{N_2}(\cos \eta)| \, d\sigma(\mathbf{y}) \int_{\|\mathcal{R}\|}^{\pi - \|\mathcal{R}\|} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t \, dt \\
& \stackrel{\text{Lemma 2.(iii)}}{\leq} 4\pi(\pi L^{-1}) \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} |v_{N_2}(\cos \eta)| \sin \eta \, d\eta \, d\phi \quad \frac{3}{2\pi} (5N_1 + 1) \\
& \leq 6(5N_1 + 1)(\pi L^{-1}) \, 2\pi \frac{1}{\pi} N_2^2 \int_0^{2\pi L^{-1}} \sin \eta \, d\eta \\
& \leq 12(5N_1 + 1) N_2^2 (\pi L^{-1}) \left[\frac{1}{2} \eta^2 \right]_0^{2\pi L^{-1}} \\
& \leq 24(5N_1 + 1) N_2^2 (\pi L^{-1})^3 \\
& \stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 1152 N_1 N_2^2 \|\mathcal{R}\|^3
\end{aligned}$$

Case (2.4):

$$\begin{aligned}
S_{2,4}^{(1)} &:= \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1, L-1}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1, L-1}} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \int_{B_{L-1}} |v_{N_2}(\cos \eta)| d\sigma(\mathbf{y}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{x}) \\
&\leq 1152 N_1 N_2^2 \|\mathcal{R}\|^3
\end{aligned}$$

Case (3):

$$\begin{aligned}
S_3^{(1)} &:= \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{\ell_1, \ell_2}} \int_{R_j} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_1, \ell_2}} |v_{N_2}(\cos \eta)| \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} |v_{N_2}(\cos \eta)| d\sigma(\mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt \sin \xi d\xi d\varphi \\
&\quad \times \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} |v_{N_2}(\cos \eta)| \sin \eta d\eta d\phi \\
&\stackrel{\sin \xi \leq 2 \sin t}{\leq} 4\pi^2 2 \sum_{\ell_1=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t dt d\xi \\
&\quad \times 2 \int_{\pi L^{-1}}^{(L-1)\pi L^{-1}} |v_{N_2}(\cos \eta)| \sin \eta d\eta \\
&\leq 16\pi^2 \sum_{\ell_1=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \int_{\xi_j^-}^{\xi} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t dt d\xi \\
&\quad \times \|v_{N_2}\|_{\infty} \int_{\pi L^{-1}}^{\pi} \sin \eta d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq 16\pi^2 2(\pi L^{-1}) \sum_{\ell_1=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t \, dt \\
&\quad \times \|v_{N_2}\|_\infty \int_{\pi L^{-1}}^{\pi} \sin \eta \, d\eta \\
&\leq 32\pi^2 (\pi L^{-1}) \int_{\|\mathcal{R}\|}^{\pi - \|\mathcal{R}\|} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin t \, dt \\
&\quad \times \frac{1}{\pi} N_2^2 [-\cos \eta]_{\pi L^{-1}}^{\pi} \\
&\leq 32\pi^2 (\pi L^{-1}) \frac{3}{2\pi} (5N_1 + 1) \frac{1}{\pi} N_2^2 2 \sin^2\left(\frac{\pi L^{-1}}{2}\right) \\
&\leq 96 (\pi L^{-1}) (5N_1 + 1) N_2^2 \left(\frac{\pi L^{-1}}{2}\right)^2 \\
&\leq 24 \left(5 + \frac{1}{N_1}\right) N_1 N_2^2 (\pi L^{-1})^3 \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 1152 N_1 N_2^2 \|\mathcal{R}\|^3.
\end{aligned}$$

All the estimates considered so far provide an estimate for the sum Σ_1 . For the second sum Σ_2 we obtain similarly

Case (1.1): Using Lemma 2 (i),(ii) we get

$$\begin{aligned}
S_{1,1}^{(2)} &= \sum_{R_j \subset B_{1,1} \cup B_{1,L-1}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta_j} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt \, d\tau(\mathbf{x}, \mathbf{y}) \\
&\stackrel{\text{Lemma 2.(ii)}}{\leq} \frac{2}{\pi} N_2^3 \|\mathcal{R}\| \frac{1}{\pi} N_1^2 \{\tau(B_{1,1}) + \tau(B_{1,L-1})\} \\
&\leq \frac{2}{\pi} N_2^3 \|\mathcal{R}\| \frac{1}{\pi} N_1^2 \\
&\quad \times \left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \int_0^{2\pi L^{-1}} + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{\pi-2\pi L^{-1}}^{\pi} \int_{\pi-2\pi L^{-1}}^{\pi} \right\} \sin \xi \sin \eta \, d\xi \, d\eta \, d\varphi \, d\phi \\
&\leq \frac{2}{\pi^2} N_2^3 \|\mathcal{R}\| N_1^2 4\pi^2 2 \left(\int_0^{2\pi L^{-1}} \sin t \, dt \right)^2 \\
&\leq 16 N_1^2 N_2^3 \|\mathcal{R}\| (\pi L^{-1})^4 \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 1024 N_1^2 N_2^3 \|\mathcal{R}\|^5.
\end{aligned}$$

Due to symmetry we have $S_{1,k}^{(2)} \leq 1024 N_1^2 N_2^3 \|\mathcal{R}\|^5$, $k = 2, 3, 4$.

Case (2.1):

$$\begin{aligned}
S_{2,1}^{(2)} &:= \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{1,\ell_2}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_2=2}^{L-2} \int_{B_{1,\ell_2}} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \int_{B_1} |v_{N_1}(\cos \xi_j)| d\sigma(\mathbf{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\sigma(\mathbf{y}) \\
&\stackrel{\text{Lemma 2.(i)}}{\leq} \|v_{N_1}\|_{\infty} \int_{-\pi}^{\pi} \int_0^{2\pi L^{-1}} \sin \xi d\xi d\varphi \\
&\quad \times \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin \eta dt d\eta d\phi \\
&\leq 4\pi N_1^2 \left[\frac{1}{2} \xi^2 \right]_0^{2\pi L^{-1}} \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin \eta dt d\eta \\
&\leq 8\pi N_1^2 (\pi L^{-1})^2 \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| 2 \sin t dt d\eta \\
&\leq 16\pi N_1^2 (\pi L^{-1})^2 2(\pi L^{-1}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin t dt \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 256\pi N_1^2 \|\mathcal{R}\|^3 \int_{\|\mathcal{R}\|}^{\pi - \|\mathcal{R}\|} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin t dt \\
&\stackrel{\text{Lemma 2.(iii)}}{\leq} 256\pi N_1^2 \|\mathcal{R}\|^3 \frac{3}{2\pi} (5N_2 + 1) \\
&\leq 256 N_1^2 N_2 \|\mathcal{R}\|^3 \frac{3}{2} \left(5 + \frac{1}{N_2} \right) \\
&\leq 2304 N_1^2 N_2 \|\mathcal{R}\|^3
\end{aligned}$$

Case (2.2): This case is similar to the case (2.1) and therefore we can shorten the calculation.

$$S_{2,2}^{(2)} := \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{L-1,\ell_2}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y})$$

$$\begin{aligned}
&\leq \sum_{\ell_2=2}^{L-2} \int_{B_{L-1, \ell_2}} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \int_{B_{L-1}} |v_{N_2}(\cos \xi_j)| d\sigma(\mathbf{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{y}) \\
&\leq 2304 N_1^2 N_2 \|\mathcal{R}\|^3
\end{aligned}$$

Case (2.3):

$$\begin{aligned}
S_{2,3}^{(2)} &:= \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1, 1}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1, 1}} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \int_{B_1} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{x}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} |v_{N_2}(\cos \xi_j)| d\sigma(\mathbf{y}) \\
&\leq 4\pi^2 \int_0^{2\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| \sin \eta dt d\eta \\
&\quad \times \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} |v_{N_2}(\cos \xi_j)| \sin \xi d\xi \\
&\stackrel{\text{Lemma 2.(ii)}}{\leq} 4\pi^2 \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \int_0^{2\pi L^{-1}} \sin \eta d\eta \|v_{N_1}\|_{\infty} 2 \int_{\pi L^{-1}}^{\pi} \sin \xi d\xi \\
&\stackrel{\text{Lemma 2.(i)}}{\leq} 16\pi N_1^3 \|\mathcal{R}\| \frac{1}{\pi} N_2^2 \left[\frac{1}{2} \eta^2 \right]_0^{2\pi L^{-1}} [-\cos \xi]_{\pi L^{-1}}^{\pi} \\
&\leq 32 N_1^3 \|\mathcal{R}\| N_2^2 (\pi L^{-1})^2 2 \sin^2\left(\frac{\pi L^{-1}}{2}\right) \\
&\leq 16 N_1^3 N_2^2 \|\mathcal{R}\| (\pi L^{-1})^4 \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 256 N_1^3 N_2^2 \|\mathcal{R}\|^5
\end{aligned}$$

Case (2.4):

$$S_{2,4}^{(2)} := \sum_{\ell_1=2}^{L-2} \sum_{R_j \subset B_{\ell_1, L-1}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y})$$

$$\begin{aligned}
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1, L-1}} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \int_{B_{L-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_1}(\cos t) \right| dt d\sigma(\mathbf{x}) \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} |v_{N_2}(\cos \xi_j)| d\sigma(\mathbf{y}) \\
&\leq 4\pi^2 \int_{\pi-2\pi L^{-1}}^{\pi} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin \eta dt d\eta \sum_{\ell_2=2}^{L-2} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} |v_{N_2}(\cos \xi_j)| \sin \xi d\xi \\
&\stackrel{\text{Lemma 2.(ii)}}{\leq} 4\pi^2 \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \int_{\pi-2\pi L^{-1}}^{\pi} \sin \eta d\eta \quad \|v_{N_1}\|_{\infty} 2 \int_{\pi L^{-1}}^{\pi} \sin \xi d\xi \\
&\leq 4\pi^2 \frac{2}{\pi} N_1^3 \|\mathcal{R}\| \int_0^{2\pi L^{-1}} \sin \eta d\eta \quad \|v_{N_1}\|_{\infty} 2 \int_{\pi L^{-1}}^{\pi} \sin \xi d\xi \\
&\stackrel{\text{Lemma 2.(i)}}{\leq} 16\pi N_1^3 \|\mathcal{R}\| \frac{1}{\pi} N_2^2 \left[\frac{1}{2} \eta^2 \right]_0^{2\pi L^{-1}} \quad [-\cos \xi]_{\pi L^{-1}}^{\pi} \\
&\leq 32 N_1^3 \|\mathcal{R}\| \frac{1}{\pi} N_2^2 (\pi L^{-1})^2 2 \sin^2\left(\frac{\pi L^{-1}}{2}\right) \\
&\leq 16 N_1^3 N_2^2 \|\mathcal{R}\| (\pi L^{-1})^4 \\
&\stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 256 N_1^3 N_2^2 \|\mathcal{R}\|^5
\end{aligned}$$

Case (3):

$$\begin{aligned}
S_3^{(2)} &:= \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \sum_{R_j \subset B_{\ell_1, \ell_2}} \int_{R_j} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_1, \ell_2}} |v_{N_1}(\cos \xi_j)| \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\tau(\mathbf{x}, \mathbf{y}) \\
&\leq \sum_{\ell_1=2}^{L-2} \int_{B_{\ell_1}} |v_{N_1}(\cos \xi_j)| d\sigma(\mathbf{x}) \sum_{\ell_2=2}^{L-2} \int_{B_{\ell_2}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt d\sigma(\mathbf{y}) \\
&\leq \sum_{\ell_1=1}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_1-1)\pi L^{-1}}^{(\ell_1+1)\pi L^{-1}} |v_{N_1}(\cos \xi_j)| \sin \xi d\xi d\varphi \\
&\quad \times \sum_{\ell_2=2}^{L-2} \int_{-\pi}^{\pi} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2+1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| dt \sin \eta d\eta d\phi
\end{aligned}$$

$$\begin{aligned}
& \begin{aligned} & \sin \xi \leq 2 \sin t \\ & \leq 4\pi^2 2 \int_{\pi L^{-1}}^{(L-1)\pi L^{-1}} |v_{N_2}(\cos \xi_j)| \sin \xi \, d\xi \\ & \quad \times 2 \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin t \, dt \, d\eta \\ & \leq 16\pi^2 \|v_{N_1}\|_{\infty} \int_{\pi L^{-1}}^{\pi} \sin \xi \, d\xi \\ & \quad \times \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}} \int_{\eta_j^-}^{\eta} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin t \, dt \, d\eta \\ & \leq 16\pi^2 \frac{1}{\pi} N_1^2 [-\cos \xi]_{\pi L^{-1}}^{\pi} 2(\pi L^{-1}) \sum_{\ell_2=2}^{L-2} \int_{(\ell_2-1)\pi L^{-1}}^{(\ell_2-1)\pi L^{-1}} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin t \, dt \, d\eta \\ & \leq 16\pi^2 \frac{1}{\pi} N_1^2 2 \sin^2\left(\frac{\pi L^{-1}}{2}\right) 2(\pi L^{-1}) 2 \int_{\|\mathcal{R}\|}^{\pi\|\mathcal{R}\|} \left| \frac{d}{dt} v_{N_2}(\cos t) \right| \sin t \, dt \\ & \leq 128\pi N_1^2 \left(\frac{\pi L^{-1}}{2}\right)^2 (\pi L^{-1}) \frac{3}{2\pi} (5N_2 + 1) \\ & \leq 16 \left(5 + \frac{1}{N_2}\right) N_1^2 N_2 (\pi L^{-1})^3 \\ & \stackrel{\pi L^{-1} \leq 2\|\mathcal{R}\|}{\leq} 768 N_1^2 N_2 \|\mathcal{R}\|^3. \end{aligned}
\end{aligned}$$

Putting all the estimates for $\Sigma^{(1)}$ and $\Sigma^{(2)}$ together we obtain the following result .

$$\Sigma \leq 8832\tilde{N}^3 \|\mathcal{R}\|^3 + 9216\tilde{N}^5 \|\mathcal{R}\|^5 \quad (5)$$

Assume $\|\mathcal{R}\| \leq \frac{\eta}{C\tilde{N}}$. Then we obtain

$$\Sigma \leq \left(8832 \frac{1}{C^3} + 9216 \frac{1}{C^5}\right) \eta \quad (6)$$

A short computation shows that for $C = 21$ the first factor in (6) is less than 1. This shows the desired result. \blacksquare

References

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